Contents

1 Introduction **1**

2 Setup **3**

3 Movement building model **5**
- 3.1 Setup 5
- 3.2 Hamiltonian equations 6
- 3.3 Variable ratios 8
- 3.4 Balanced growth equations 10
- 3.5 Balanced growth path derivation 11
- 3.6 Checking the transversality condition 11
- 3.7 Results 12

4 Exact results and numerical simulations **17**
- 4.1 Exact results 17
- 4.2 Numerical simulations 19

5 Conclusions and discussion **40**

6 References **43**

A Minimal logarithmic model **44**
- A.1 Setup 44
- A.2 Optimal path derivation 45
- A.3 Checking the transversality condition 46
- A.4 Results and interpretation 47

B Some other numerical simulation outputs **48**

Labor, Capital, and the Optimal Growth of Social Movements

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WORK IN PROGRESS

1 Introduction

Social movements such as "Effective Altruism" face the problem of optimal allocation of resources across time in order to maximize their desired impact. Much like states and other entities considered in the literature since (Ramsey, 1928) [1], they have the option to invest in order to give more later. However, unlike states, where population dynamics are usually considered exogenous, such agents also have the option of recruiting like-minded associates through movement building. For example, Bill Gates can recruit other ultra-rich people through the Giving Pledge, aspiring effective altruists can likewise spread their ideas, etc.

This paper models the optimal allocation of capital for a social movement between direct spending, investment, and movement building, as well as the optimal allocation of labor between direct workers, money earners, and movement builders. This research direction follows in the footsteps of (Trammell, 2020) [2], which considers a different facet of a related problem: the dynamics for a philanthropic funder who aims to provide public goods while having a lower discount rate than less patient partners.

The outline of this paper is as follows: §2 presents the mathematical toolset and nomenclature used to find solutions for the optimal path problems, namely the Hamiltonian.

§3 considers a social movement which starts out with a certain amount of money and a certain number of movement participants. This movement must then decide where to allocate their capital and labor. We work out some useful properties of the optimal solution, as well as its long-term balanced growth rates.

§4 outlines some results which allow for the numerical simulation of the evolution of the model starting from some initial conditions, and provides some analysis of that evolution. §5 outlines implications and conclusions.

2 Setup

We're interested in the following general maximization problem

$$V(\vec{\alpha(t)}) = \max_{\vec{\alpha(t)}} \int_0^\infty e^{-\rho t} \cdot U(\vec{x(t)}, \vec{\alpha(t)})dt \tag{1}$$

Subject to

$$\dot{\vec{x}} = f(t, \vec{x(t)}, \vec{\alpha(t)}) \tag{2}$$

$$x_i \geq 0 \tag{3}$$

$$\vec{x}(0) = \vec{x_0} \tag{4}$$

Where the variables stand for:

1. $\rho =$ Discount rate, perhaps value drift, risk of theft, etc.
2. $\vec{x} = \begin{bmatrix} x_1(t) \\ x_3(t) \\ ... \end{bmatrix} = \begin{bmatrix} \text{Capital at time } t \\ \text{Movement size at time } t \\ ... \end{bmatrix}$
3. $\vec{\alpha} = \begin{bmatrix} \alpha_1(t) \\ \alpha_3(t) \\ ... \end{bmatrix} = \begin{bmatrix} \text{Investment into altruistic projects at time } t \\ \text{Investment into movement building at time } t \\ .. \end{bmatrix}$
4. $f =$ Function relating $\vec{x}$ and $\vec{\alpha}$
5. $U =$ The utility function. A function from capital, information and current spending and other resources into altruistic impact

To find the optimal allocation path, we can define the current value Hamiltonian[1]:

$$H := U(\vec{x(t)}, \vec{\alpha(t)}) + \vec{\mu(t)}^{(T)} \cdot \dot{\vec{x}} \tag{5}$$

$$H = U(\vec{x(t)}, \vec{\alpha(t)}) + \mu_1(t) \cdot \dot{x_1}(t) + \mu_3(t) \cdot \dot{x_3}(t) \tag{6}$$

The solution, that is, the optimal allocation path, will be given by the following constraints on the Hamiltonian:

[1]Casual readers have repeatedly thought this referred to the Hamiltonian operator in quantum mechanics, or to one of the many other objects refered to as a "Hamiltonian". We note that we refer to the Hamiltonian in optimal control theory.

$$\frac{\partial H}{\partial \vec{\alpha}} = 0 \tag{7}$$

$$-\frac{\partial H}{\partial \vec{x}} = \dot{\vec{\mu}} - \rho\vec{\mu} \tag{8}$$

$$-\frac{\partial H}{\partial \vec{\mu}} = \dot{\vec{x}} \tag{9}$$

Further, the solution must conform to the following transversality condition:

$$\lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot (\vec{x} \cdot \vec{\mu}) = \vec{0} \tag{10}$$

or, in scalar form

$$\lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot x_i \cdot \mu_i = 0 \tag{11}$$

We will drop the vector references and references to time from now on, while keeping them in mind. The mathematically sophisticated reader without previous knowlesdge of optimal control theory who is looking for a short introduction to Hamiltonians, and to why they provide a solution to our maximization problem, is welcome to consult (Kurat 2013) [4]. For a minimal model to familiarize oneself with the concepts, consult either the basic model in (Trammell, 2020) [2], or a rather minimal model in Appendix §A.

3 Movement building model

3.1 Setup

The variables under consideration are:

1. x_1, total capital, and α_1, spending on direct work on a given instant.
2. x_3, total movement size (labor), and α_3, the money spent on movement building on a given instant.
3. $\sigma_1, \sigma_2, \sigma_3$: the fraction of the movement which works respectively on direct work, money-making, and movement building. $\sigma_1 + \sigma_2 + \sigma_3 = 1$, so well substitute $\sigma_2 = 1 - \sigma_1 - \sigma_3$ throughout.
4. $w_3 \cdot exp\{\gamma_1 t\}$: wages rising with economic growth, and $\beta_3 \cdot exp\{\gamma_3 t\}$: the changing difficulty of recruiting movement participants. γ_3 might be hypothesized to be negative, given that economic growth provides better outside options, but empirically seems to be positive. For simplicity, we will consider these rates —γ_1 and γ_3— to be exogenous.
5. δ_3: movement building returns to scale

We are maximizing:

$$V(\vec{\alpha(t)}) = \max_{\vec{\alpha(t)}} \int_0^\infty e^{-\rho t} \cdot U(\vec{x(t)}, \vec{\alpha(t)}) dt \tag{12}$$

For utility and laws of motion:

$$U(x, \alpha) = \frac{(\alpha_1^{\lambda_1} (\sigma_1 x_3)^{1-\lambda_1})^{1-\eta}}{1-\eta} \tag{13}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} r_1 x_1 - \alpha_1 - \alpha_3 + x_3 \cdot w_3 \cdot exp\{\gamma_1 t\} \cdot (1 - \sigma_1 - \sigma_3) \\ \beta_3 \cdot exp\{\gamma_3 t\} \cdot (\alpha_3^{\lambda_3} \cdot (\sigma_3 x_3)^{1-\lambda_3})^{\delta_3} \end{bmatrix} \tag{14}$$

under the constraints that

$$x_3 \geq 0 \wedge \alpha_i \geq 0 \wedge \sigma_1 + \sigma_3 \leq 1 \tag{15}$$

With the Hamiltonian standing at:

$$H = U + \mu_1 \cdot \dot{x}_1 + \mu_3 \cdot \dot{x}_3 \tag{16}$$

$$\begin{aligned} H = & \frac{(\alpha_1^{\lambda_1}(\sigma_1 x_3)^{1-\lambda_1})^{1-\eta}}{1-\eta} \\ & + \mu_1 \cdot (r_1 x_1 - \alpha_1 - \alpha_3 + x_3 \cdot w_3 \cdot exp\{\gamma_1 t\} \cdot (1 - \sigma_1 - \sigma_3)) \\ & + \mu_3 \cdot (\beta_3 \cdot exp\{\gamma_3 t\} \cdot (\alpha_3^{\lambda_3} \cdot (\sigma_3 x_3)^{1-\lambda_3})^{\delta_3}) \end{aligned} \tag{17}$$

and the transversality condition same as it ever was:

$$\lim_{t \to \infty} exp\{-\rho \cdot t\} \cdot x_i \cdot \mu_i = 0 \tag{18}$$

For convenience, $F_3 := \beta_3 \cdot (\alpha_3^{\lambda_3} \cdot (\sigma_3 x_3)^{1-\lambda_3})^{\delta_3}$. Note that $F_3 = \dot{x}_3$

3.2 Hamiltonian equations

3.2.1 $\frac{\partial H}{\partial \alpha_1} = 0$

$$(1-\eta) \cdot \lambda_1 \cdot \frac{U}{\alpha_1} - \mu_1 = 0 \tag{19}$$

$$\mu_1 = (1-\eta)\lambda_1 \cdot \frac{U}{\alpha_1} \tag{20}$$

3.2.2 $\frac{\partial H}{\partial \alpha_3} = 0$

$$\mu_3 \cdot \delta_3 \lambda_3 \cdot \frac{F_3}{\alpha_3} - \mu_1 = 0 \tag{21}$$

$$\mu_1 = \mu_3 \cdot \delta_3 \cdot \lambda_3 \cdot \frac{F_3}{\alpha_3} \tag{22}$$

3.2.3 $\frac{\partial H}{\partial \sigma_1} = 0$

$$(1-\eta)(1-\lambda_1) \cdot \frac{U}{\sigma_1} - \mu_1 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\} = 0 \tag{23}$$

$$\mu_1 = \frac{(1-\eta)(1-\lambda_1)}{w_3} \cdot \frac{U}{\sigma_1 \cdot x_3 \cdot exp\{\gamma_1 t\}} \tag{24}$$

3.2.4 $\frac{\partial H}{\partial \sigma_3} = 0$

$$-\mu_1 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\} + \mu_3 \cdot \delta_3 (1-\lambda_3) \cdot \frac{F_3}{\sigma_3} = 0 \tag{25}$$

$$\mu_1 = \mu_3 \cdot \frac{\delta_3 \cdot (1-\lambda_3)}{w_3} \cdot \frac{F_3}{\sigma_3 \cdot x_3 \cdot exp\{\gamma_1 t\}} \tag{26}$$

3.2.5 $\frac{\partial H}{\partial x_1} = \rho\mu_1 - \dot{\mu}_1$

$$\mu_1 \cdot r_1 = \rho\mu_1 - \dot{\mu}_1 \tag{27}$$

$$\mu_1 = k_1 \cdot exp\{(\rho - r_1)t\} \tag{28}$$

3.2.6 $\frac{\partial H}{\partial x_3} = \rho\mu_3 - \dot{\mu}_3$

$$\begin{aligned} \rho\mu_3 - \dot{\mu}_3 = \mu_3 \cdot (\rho - g_{\mu_3}) = (1-\eta) \cdot (1-\lambda_1) \cdot \frac{U}{x_3} \\ + \mu_1 \cdot w_3 \cdot exp\{\gamma_1 t\} \cdot (1 - \sigma_1 - \sigma_3) \\ + \mu_3 \cdot (1-\lambda_3) \cdot \delta_3 \cdot \frac{F_3}{x_3} \end{aligned} \tag{29}$$

Through several manipulations of (29), in particular by substituting $(1-\eta) \cdot (1-\lambda_1) \cdot U$ from (24) and $(1-\lambda_1) \cdot \delta_3 \cdot F_3 \cdot \mu_3$ from (26), we arrive at:

$$\mu_3 \cdot (\rho - g_{\mu_3}) = \mu_1 \cdot w_3 \cdot exp\{\gamma_1 t\} \tag{30}$$

This produces the growth equation

$$g_{\mu_1} = g_{\mu_3} + g_{x_3} - \gamma_1 \tag{31}$$

3.2.7 Summary

$$\mu_1 = (1-\eta)\lambda_1 \cdot \frac{U}{\alpha_1} \tag{32}$$

$$\mu_1 = \mu_3 \cdot \delta_3 \cdot \lambda_3 \cdot \frac{F_3}{\alpha_3} \tag{33}$$

$$\mu_1 = \frac{(1-\eta)(1-\lambda_1)}{w_3} \cdot \frac{U}{\sigma_1 \cdot exp\{\gamma_1 t\}} \tag{34}$$

$$\mu_1 = \mu_3 \cdot \frac{\delta_3 \cdot (1-\lambda_3)}{w_3} \cdot \frac{F_3}{\sigma_3 \cdot exp\{\gamma_1 t\}} \tag{35}$$

$$\mu_1 = k_1 \cdot exp\{(\rho - r_1)t\} \tag{36}$$

$$\mu_3 \cdot (\rho - g_{\mu_3}) = \mu_1 \cdot w_3 \cdot exp\{\gamma_1 t\} \tag{37}$$

3.3 Variable ratios

By dividing (32) by (34) and (33) by (35), we conclude that:

$$\frac{\lambda_1}{\alpha_1} = \frac{1-\lambda_1}{\sigma_1 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\}} \tag{38}$$

$$\frac{\lambda_3}{\alpha_3} = \frac{1-\lambda_3}{\sigma_3 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\}} \tag{39}$$

and hence

$$\frac{(1-\lambda_1)}{\lambda_1} \cdot \frac{\alpha_1}{\sigma_1} = \frac{(1-\lambda_3)}{\lambda_3} \cdot \frac{\alpha_3}{\sigma_3} \tag{40}$$

or, even more simply,

$$\frac{\alpha_1}{\sigma_1} \cdot \frac{\sigma_3}{\alpha_3} = \text{constant} \tag{41}$$

respectively

$$\frac{\alpha_1}{\sigma_1 \cdot x_3} \cdot \frac{\sigma_3 \cdot x_3}{\alpha_3} = \text{constant} \tag{42}$$

Now, (40) is important enough that we will rederive it from the Euler equations, that is, just from the constraint that on the optimal path, the marginal value of moving people and funds around should be equal to 0. In particular,

$$\frac{\partial U}{\partial \$} = \frac{\partial U}{\partial people} \cdot \frac{\partial people}{\partial \$}_{\text{bought out of money-making}} \tag{43}$$

$$\frac{\partial people}{\partial \$}_{\text{through movement building}} = \frac{\partial people}{\partial people} \cdot \frac{\partial people}{\partial \$}_{\text{bought out of money-making}} \tag{44}$$

Equation (43) reads as "the *marginal* money-maker should produce as much value by making money and directly donating their earnings as by working directly." Equation (43) reads as "the *marginal* money-maker should create as many movement participants by making money and donating their earnings to movement building as by working on movement building themselves." Otherwise, we could move direct workers or movement builders towards money-making, or vice-versa.

From (14) and (16), the model definition, these two equations develop into:

$$\lambda_1 \cdot (1-\eta) \cdot \frac{U}{\alpha_1} = \left((1-\lambda_1) \cdot (1-\eta) \cdot \frac{U}{\sigma_1 \cdot x_3} \right) \cdot \left(\frac{1}{w_3 \cdot exp\{\gamma_1 \cdot t\}} \right) \tag{45}$$

$$\lambda_3 \cdot \delta_3 \cdot \frac{F_3}{\alpha_3} = \left((1-\lambda_3) \cdot \delta_3 \cdot \frac{F_3}{\sigma_3 \cdot x_3} \right) \cdot \left(\frac{1}{w_3 \cdot exp\{\gamma_1 \cdot t\}} \right) \tag{46}$$

Which simplify into

$$\frac{\lambda_1}{\alpha_1} = \frac{1-\lambda_1}{\sigma_1 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\}} \tag{47}$$

$$\frac{\lambda_3}{\alpha_3} = \frac{1-\lambda_3}{\sigma_3 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\}} \tag{48}$$

i.e., (38) and (39), from which (40) follows:

$$\frac{(1-\lambda_1)}{\lambda_1} \cdot \frac{\alpha_1}{\sigma_1} = \frac{(1-\lambda_3)}{\lambda_3} \cdot \frac{\alpha_3}{\sigma_3} \tag{49}$$

We can understand this equation as a convenient necessary but not sufficient heuristic, such that a spending schedule which doesn't satisfy it suffers from the affliction that, insofar as the model is accurate enough, one would be able to obtain a better outcome by redistributing people and funds around.

3.4 Balanced growth equations

3.4.1 Balanced growth equations I

$$g_{\mu_1} = g_U - g_{\alpha_1} \tag{50}$$

$$g_{\mu_1} = g_{\mu_3} + g_{F_3} - g_{\alpha_3} \tag{51}$$

$$g_{\mu_1} = g_U - g_{\sigma_1} - g_{x_3} - \gamma_1 \tag{52}$$

$$g_{\mu_1} = g_{\mu_3} + g_{F_3} - g_{\sigma_3} - g_{x_3} - \gamma_1 \tag{53}$$

$$g_{\mu_1} = (\rho - r_1) \tag{54}$$

$$g_{\mu_1} = g_{\mu_3} - \gamma_1 \tag{55}$$

$$g_{x_3} = g_{F_3} = \gamma_3 + \delta_3 \cdot \Big(\lambda_3 \cdot g_{\alpha_3} + (1 - \lambda_3) \cdot (g_{\sigma_3} + g_{x_3})\Big) \tag{56}$$

3.4.2 Balanced growth equations II

Some simple simplifications. (62) is derived from (53) + (55) + ($g_{x_3} = g_{F_3}$).

$$g_{\alpha_1} = g_{\sigma_1} + g_{x_3} + \gamma_1 \tag{57}$$

$$g_{\mu_1} = g_U - g_{\alpha_1} \tag{58}$$

$$g_{\alpha_3} = g_{\sigma_3} + g_{x_3} + \gamma_1 \tag{59}$$

$$g_{\mu_1} = g_{\mu_3} + g_{F_3} - g_{\alpha_3} \tag{60}$$

$$g_{\mu_1} = \rho - r_1 \tag{61}$$

$$g_{\sigma_3} = 0 \tag{62}$$

$$g_{x_3} = g_{F_3} = \gamma_3 + \delta_3 \cdot \Big(\lambda_3 \cdot g_{\alpha_3} + (1 - \lambda_3) \cdot (g_{\sigma_3} + g_{x_3})\Big) \tag{63}$$

3.5 Balanced growth path derivation

From this we can simply derive g_{x_3}, by substituting (59) and (62) in (63)

$$g_{x_3} = \frac{\gamma_3 + \delta_3 \lambda_3 \gamma_1}{1 - \delta_3} \tag{64}$$

And from that g_{α_3}, by substituting (64) back in (59)

$$g_{\alpha_3} = g_{x_3} + \gamma_1 = \frac{\gamma_3 + \delta_3 \lambda_3 \gamma_1}{1 - \delta_3} + \gamma_1 \tag{65}$$

Similarly, from (57), (58) and (64), we can derive g_{α_1} and g_{σ_1}:

$$g_{\alpha_1} = \frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta} \cdot \gamma_1 \tag{66}$$

$$g_{\sigma_1} = \frac{r - \rho}{\eta} - \left(\frac{(1-\eta)(1-\lambda_1)}{\eta} + 1 \right) \cdot \gamma_1 - g_{x_3} \tag{67}$$

Note that this solution is only valid where $g_{\sigma_1} \leq 0$.

Note also that $g_{\alpha_1} \leq g_{\alpha_3}$. Proof: $g_{\alpha_1} = g_{\sigma_1} + g_{x_3} + \gamma_1$, and $g_{\alpha_3} = \cancel{g_{\sigma_3}} + g_{x_3} + \gamma_1$. Hence $g_{\alpha_1} = g_{\sigma_1} + g_{\alpha_3} \wedge g_{\sigma_1} \leq 0 \implies g_{\alpha_1} \leq g_{\alpha_3}$.

We can also derive x_1.

$$\dot{x}_1 = r_1 x_1 - \alpha_1 - \alpha_3 + x_3 \cdot w_3 \cdot exp\{\gamma_1 t\} \cdot (1 - \sigma_1 - \sigma_3) \tag{68}$$

$$x_1 = a \cdot exp\{r_1 \cdot t\} + b \cdot exp\{g_{\alpha_1} \cdot t\} + c \cdot exp\{g_{\alpha_3} \cdot t\} \tag{69}$$

3.6 Checking the transversality condition

The variables we need follow. We get μ_3 from (53) + $(g_{F_3} = g_{x_3})$ + $(g_{\sigma_3} = 0)$

$$\mu_1 = k_1 \cdot exp\{(\rho - r_1) \cdot t\} \tag{70}$$

$$\mu_3 = exp\left\{ \Big((\rho - r_1) + \gamma_1 \Big) \cdot t \right\} \tag{71}$$

$$x_1 = a \cdot exp\{r_1 \cdot t\} + b \cdot exp\{g_{\alpha_1} \cdot t\} + c \cdot exp\{g_{\alpha_3} \cdot t\} \tag{72}$$

$$x_3 = exp\left\{\frac{\gamma_3 + \delta_3\lambda_3\gamma_1}{1-\delta_3}\cdot t\right\} \tag{73}$$

The transversality condition is

$$\lim_{t\to\infty} exp\{-\rho\cdot t\}\cdot x_i\cdot \mu_i = 0 \tag{74}$$

For $i = 1$, this implies $a = 0$, $g_{\alpha_1} < r_1$, $g_{\alpha_3} < r_1$. For $\rho \approx 0.005, \gamma_1 \approx 0.05, \gamma_3 \approx 0.01, r_1 \approx 0.06, \lambda_1 \approx 0.5$, this implies $\eta \gtrsim 0.86$.

For $i = 3$, the transversality condition is satisfied when:

$$-\rho + (\rho - r_1 + \gamma_1) + \frac{\gamma_3 + \delta_3\cdot\lambda_3\cdot\gamma_1}{1-\delta_3} < 0 \tag{75}$$

i.e.,

$$\gamma_1 + \frac{\gamma_3 + \delta_3\cdot\lambda_3\cdot\gamma_1}{1-\delta_3} < r_1 \tag{76}$$

or, alternatively,

$$g_{\alpha_3} = gx_1 + \gamma_1 < r_1 \tag{77}$$

For $\lambda_3 \approx 0.5, r_1 \approx 0.06, \gamma_1 \approx 0.03, \gamma_3 \approx 0.01$, this implies that either $\delta_3 \lesssim 0.044$ or $1 < \delta_3$. For $\gamma_1 \approx 0.02$, this changes to $-1 < \delta_3 \lesssim 0.6$ or $1 < \delta_3$.

3.7 Results

3.7.1 Balanced growth rates

$$g_{x_3} = \frac{\gamma_3 + \delta_3\lambda_3\gamma_1}{1-\delta_3} \tag{78}$$

$$g_{\alpha_3} = g_{x_3} + \gamma_1 = \left[\frac{\gamma_3 + \delta_3\lambda_3\gamma_1}{1-\delta_3}\right] + \gamma_1 \tag{79}$$

$$g_{\sigma_3} = 0 \tag{80}$$

$$g_{\alpha_1} = \frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta}\cdot\gamma_1 \tag{81}$$

$$g_{\sigma_1} = \frac{r - \rho}{\eta} - \left(\frac{(1-\eta)(1-\lambda_1)}{\eta} + 1\right)\cdot\gamma_1 - g_{x_3} \tag{82}$$

3.7.2 Asymptotic Ponzi

Per (40):

$$\frac{(1-\lambda_1)}{\lambda_1} \cdot \frac{\alpha_1}{\sigma_1} = \frac{(1-\lambda_3)}{\lambda_3} \cdot \frac{\alpha_3}{\sigma_3} \tag{83}$$

Per results on the previous section, $g_{\sigma_3} = 0$, and $g_{\sigma_1} \leq 0$. In particular, $g_{\sigma_1} < 0$ unless we're on the knife edge case where

$$\frac{r-\rho}{\eta} - \left(\frac{(1-\eta)(1-\lambda_1)}{\eta} + 1\right) \cdot \gamma_1 = \frac{\gamma_3 + \delta_3\lambda_3\gamma_1}{1-\delta_3} \tag{84}$$

So, unless our variables are on that knife edge case, σ_3 tends to a constant and $\sigma_1 \to 0$, and hence per (40), $g_{\alpha_3} > g_{\alpha_1}$

$$\frac{\sigma_1}{\sigma_1 + \sigma_3} \to 0 \tag{85}$$

$$\frac{\alpha_1}{\alpha_1 + \alpha_3} \to 0 \tag{86}$$

This is reminiscent of a Ponzi scheme or of a multi-level-marketing scheme, because in the limit, most participants don't do direct-work either. In section §4 we will notice that this behavior may hold in the limit, but doesn't hold in the near-term.

3.7.3 Example values: $\eta = 1.1, \gamma_1 = 0.03, \delta_3 = 0.44$

$$\begin{cases} \eta = \text{Elastic of spending} = 1.1 \\ \rho = \text{Hazard rate} = 0.005 = 0.5\% \\ r_1 = \text{Returns above inflation} = 0.06 = 6\% \\ \gamma_1 = \text{Change in participant contributions} = 0.03 = 3\% \\ \gamma_3 = \text{Change in the difficulty of recruiting} = 0.01 = 1\% \\ w_3 = \text{Average participant contribution per unit of time} = 0.5 \\ \beta_3 = \text{Constant inversely proportional to difficulty of recruiting} = 1,000 \\ \lambda_1 = \text{Coub-Douglas elasticity of direct work and direct spending} = 0.5 \\ \lambda_3 = \text{Coub-Douglas elasticity of movement building} = 0.5 \\ \delta_3 = \text{Elasticity of movement growth} = 0.44 \end{cases} \tag{87}$$

$$
\begin{aligned}
g_{x_3} &= \frac{\gamma_3 + \delta_3 \lambda_3 \gamma_1}{1 - \delta_3} \\
&= \frac{0.01 + 0.5 \cdot 0.44 \cdot 0.03}{1 - 0.44} \\
&= 0.0296 = 2.96\%
\end{aligned} \tag{88}
$$

$$
\begin{aligned}
g_{\alpha_3} &= g_{\sigma_3} + g_{x_3} + \gamma_1 = 0 + g_{x_3} + \gamma_1 \\
&= 0.0296 + 0.03 \\
&= 0.0596 = 5.96\%
\end{aligned} \tag{89}
$$

$$
\begin{aligned}
g_{\alpha_1} &= \frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta} \cdot \gamma_1 \\
&= \frac{0.06 - 0.005}{1.1} - \frac{(1-1.1)(1-0.5)}{1.1} \cdot 0.03 \\
&\approx 0.05136 = 5.136\%
\end{aligned} \tag{90}
$$

$$
\begin{aligned}
g_{\sigma_1} &= g_{\alpha_1} - g_{x_3} - \gamma_1 \\
&= 0.05136 - 0.0296 - 0.03 \\
&= -0.00824 = -0.824\%
\end{aligned} \tag{91}
$$

3.7.4 Example values: $\eta = 0.9, \gamma_1 = 0.03, \delta_3 = 0.44$

$$
\begin{cases}
\eta = \text{Elasticy of spending} = 0.9 \\
\rho = \text{Hazard rate} = 0.005 = 0.5\% \\
r_1 = \text{Returns above inflation} = 0.06 = 6\% \\
\gamma_1 = \text{Change in participant contributions} = 0.03 = 3\% \\
\gamma_3 = \text{Change in the difficulty of recruiting} = 0.01 = 1\% \\
w_3 = \text{Average participant contribution per unit of time} = 0.5 \\
\beta_3 = \text{Constant inversely proportional to difficulty of recruiting} = 1,000 \\
\lambda_1 = \text{Coub-Douglas elasticity of direct work and direct spending} = 0.5 \\
\lambda_3 = \text{Coub-Douglas elasticity of movement building} = 0.5 \\
\delta_3 = \text{Elasticity of movement growth} = 0.44
\end{cases} \tag{92}
$$

$$
\begin{aligned}
g_{x_3} &= \frac{\gamma_3 + \delta_3 \lambda_3 \gamma_1}{1 - \delta_3} \\
&= \frac{0.01 + 0.5 \cdot 0.44 \cdot 0.03}{1 - 0.44} \\
&= 0.0296 = 2.96\%
\end{aligned} \tag{93}
$$

$$
\begin{aligned}
g_{\alpha_3} &= g_{\sigma_3} + g_{x_3} + \gamma_1 = 0 + g_{x_3} + \gamma_1 \\
&= 0.0296 + 0.03 \\
&= 0.0596 = 5.96\%
\end{aligned} \tag{94}
$$

$$
\begin{aligned}
g_{\alpha_1} &= \frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta} \cdot \gamma_1 \\
&= \frac{0.06 - 0.005}{0.9} - \frac{(1-0.9)(1-0.5)}{0.9} \cdot 0.03 \\
&\approx 0.0594 = 5.94\%
\end{aligned} \tag{95}
$$

$$
\begin{aligned}
g_{\sigma_1} &= g_{\alpha_1} - g_{x_3} - \gamma_1 \\
&= 0.0594 - 0.0296 - 0.03 \\
&= -0.0002 = -0.02\%
\end{aligned} \tag{96}
$$

3.7.5 Comparison with a rule of thumb allocation

For $\eta = 1.1$. Take a rule of thumb allocation, where $\sigma_1 = \sigma_3 = 0.5$, and the movement spends 1% of its capital per year, which then grows at 5% per year (i.e., $g_{\alpha_1} = g_{\alpha_3} = g_{x_1} = 0.05$).

Let $\lambda_1 = \lambda_3 = 0.5$, and in general let all the variables be as in the $\eta = 1.1$ example. Then the growth rate for x_3 is:

$$g_{x_3} = \gamma_3 + \delta_3 \cdot (\lambda_3 \cdot g_{\alpha_3} + (1 - \lambda_3) \cdot (g_{\sigma_3} + g_{x_3})) \tag{97}$$

$$g_{x_3} = 0.01 + 0.5 \cdot (0.5 \cdot 0.05 + 0.44 \cdot (0 + g_{x_3})) \tag{98}$$

$$g_{x_3} = 0.0288462 \approx 0.0288 \tag{99}$$

Then consider the growth of U

$$U(x,\alpha) = \frac{(\alpha_1^{\lambda_1}(\sigma_1 x_3)^{1-\lambda_1})^{1-\eta}}{1-\eta} \tag{100}$$

$$g_U = (1-\eta)\cdot(\lambda_1 \cdot g_{\alpha_1} + (1-\lambda_1)\cdot(g_{\sigma_1} + g_{x_3})) \tag{101}$$

$$g_U = (1-1.1)\cdot(0.5\cdot 0.05 + (1-0.5)\cdot(0+0.0288)) = -0.00394 \tag{102}$$

In contrast, in the example, that growth is equal to:

$$g_U = (1-1.1)\cdot(0.5\cdot 0.0594 + (1-0.5)\cdot(-0.00824+0.0296)) \approx -0.004038 \tag{103}$$

Note that when $\eta > 1$, the utility term is always negative, and thus a faster decrease is preferable.

For $\eta = 0.9$. Using the same reasoning as before, for the rule of thumb:

$$g_{x_3} \approx 0.0288 \tag{104}$$

$$g_U = (1-0.9)\cdot(0.5\cdot 0.05 + (1-0.5)\cdot(0+0.0288)) = 0.00394 \tag{105}$$

In comparison with the optimal path:

$$g_U = (1-0.9)\cdot(0.5\cdot 0.0594 + (1-0.5)\cdot(-0.0002+0.0296)) = 0.00444 \tag{106}$$

Note that when $\eta < 1$, the utility term is positive, and so higher growth in utility is preferable.

4 Exact results and numerical simulations

4.1 Exact results

In this section, through the previous equations, we derive a more or less explicit formula for α_1 and α_3. Using that, determine the form of σ_1 and σ_3, and having these, we derive the instantaneous change in x_1 and x_3, and this is already enough for numerical simulations.

To derive α_1, we will make use of the following equations: (32), (36) and (38)

$$\mu_1 = (1-\eta)\lambda_1 \cdot \frac{U}{\alpha_1} \tag{107}$$

$$\mu_1 = k_1 \cdot exp\{(\rho - r_1)t\} \tag{108}$$

$$\frac{\lambda_1}{\alpha_1} = \frac{1-\lambda_1}{\sigma_1 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\}} \tag{109}$$

Expanding the full form of U per (13) on (107):

$$\mu_1 = \lambda_1 \cdot \frac{(\alpha_1^{\lambda_1}(\sigma_1 x_3)^{1-\lambda_1})^{1-\eta}}{\alpha_1} \tag{110}$$

and replacing $\sigma_1 \cdot x_3$ on (109) from (108):

$$\mu_1 = \lambda_1 \cdot \frac{\left(\alpha_1^{\lambda_1} \cdot \left(\frac{1-\lambda_1}{\lambda_1} \cdot \frac{\alpha_1}{w_3 \cdot exp\{\gamma_1 t\}}\right)^{1-\lambda_1}\right)^{1-\eta}}{\alpha_1} \tag{111}$$

$$\mu_1 = \lambda_1 \cdot \frac{\alpha_1^{(1-\eta)}}{\alpha_1} \cdot \left(\frac{1-\lambda_1}{\lambda_1 \cdot w_3 \cdot exp\{\gamma_1 t\}}\right)^{(1-\lambda_1)(1-\eta)} \tag{112}$$

$$\alpha_1^{\eta} = \frac{\lambda_1}{\mu_1} \cdot \left(\frac{1-\lambda_1}{\lambda_1 \cdot w_3 \cdot exp\{\gamma_1 t\}}\right)^{(1-\lambda_1)(1-\eta)} \tag{113}$$

$$\alpha_1^{\eta} = \frac{\lambda_1}{k_1 \cdot exp\{(\rho - r_1)t\}} \cdot \left(\frac{1-\lambda_1}{\lambda_1 \cdot w_3 \cdot exp\{\gamma_1 t\}}\right)^{(1-\lambda_1)(1-\eta)} \tag{114}$$

Note how this is consistent with (66):

$$g_{\alpha_1} = \frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta} \cdot \gamma_1 \tag{115}$$

We can derive α_3 in a similar manner, starting from (33), (39) and (37)

$$\mu_1 = \mu_3 \cdot \delta_3 \cdot \lambda_3 \cdot \frac{F_3}{\alpha_3} \tag{116}$$

$$\frac{\lambda_3}{\alpha_3} = \frac{1-\lambda_3}{\sigma_3 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\}} \tag{117}$$

$$\mu_3 \cdot (\rho - g_{\mu_3}) = \mu_1 \cdot w_3 \cdot exp\{\gamma_1 t\} \tag{118}$$

We expand F_3 on (119) per (14) and divide by μ_3:

$$\frac{\mu_1}{\mu_3} = \delta_3 \cdot \lambda_3 \cdot \frac{\beta_3 \cdot exp\{\gamma_3 t\} \cdot (\alpha_3^{\lambda_3} \cdot (\sigma_3 x_3)^{1-\lambda_3})^{\delta_3}}{\alpha_3} \tag{119}$$

We simplify μ_1/μ_3 per (118), replace $\sigma_1 x_3$ per (117), and substitute $g_{\mu_3} = \rho - r_1 + \gamma_1$

$$\frac{\rho - g_{\mu_3}}{w_3 \cdot exp\{\gamma_1 \cdot t\}} = \delta_3 \cdot \lambda_3 \cdot \beta_3 \cdot exp\{\gamma_3 t\} \cdot \frac{\left(\alpha_3^{\lambda_3} \cdot \left(\frac{1-\lambda_3}{\lambda_3} \cdot \frac{\alpha_3}{w_3 \cdot exp\{\gamma_1 t\}}\right)^{1-\lambda_3}\right)^{\delta_3}}{\alpha_3} \tag{120}$$

$$\frac{\rho - (\rho - r_1 + \gamma_1)}{w_3 \cdot exp\{\gamma_1 \cdot t\}} = \delta_3 \cdot \lambda_3 \cdot \beta_3 \cdot exp\{\gamma_3 t\} \cdot \frac{\alpha_3^{\delta_3}}{\alpha_3} \cdot \left(\frac{1-\lambda_3}{\lambda_3 \cdot w_3 \cdot exp\{\gamma_1 t\}} \cdot\right)^{\delta_3 \cdot (1-\lambda_3)} \tag{121}$$

$$\alpha_3^{1-\delta_3} = \frac{w_3 \cdot exp\{\gamma_1 \cdot t\}}{r_1 - \gamma_1} \cdot \delta_3 \cdot \lambda_3 \cdot \beta_3 \cdot exp\{\gamma_3 t\} \cdot \left(\frac{1-\lambda_3}{\lambda_3 \cdot w_3 \cdot exp\{\gamma_1 t\}} \cdot\right)^{\delta_3 \cdot (1-\lambda_3)} \tag{122}$$

4.2 Numerical simulations

4.2.1 Strategy

We have determined the value of α_i at all times (up to a constant k_1), as well as α_3. Now suppose we knew x_1 and x_3 at some point, for example at the present time t_0, i.e., $x_1(t_0), x_3(t_0)$. Then, we could also figure out $\sigma_i(t_0)$, per (38) and (39):

$$\sigma_i(t_0) = \frac{1-\lambda_i}{\lambda_i} \cdot \frac{\alpha_i(t_0)}{x_3(t_0) \cdot w_3 \cdot exp\{\gamma_1 t_0\}} \tag{123}$$

Using $\alpha_1(t_0), \alpha_3(t_0), \sigma_1(t_0), \sigma_3(t_0), x_1(t_0), x_3(t_0)$ we can approximate the derivative, or instantaneous change of the state variables, $\dot{x}_1(t_0), \dot{x}_3(t_0)$ per their law of motion (14), and then approximate $x_i(t_0 \pm \epsilon) = x_i(t_0) \pm \epsilon \cdot \dot{x}_i(t_0)$. Our general approach to generate numerical approximations will be to use this approximation.

The method in which we start with the values at some initial point in time and then extrapolate them into the future is known as forward shooting. In contrast, the method in which we try to guess some final points in the future which, when extrapolated into the past hit our initial conditions is known as reverse shooting. Reverse shooting is known for being more stable, but in practice we don't notice much of a difference.

The code, in R, is based on previous Matlab code by Leopold Aschenbrenner, whose code is itself based on code by Charles Jones. Aschenbrenner's code was up to academic standards, so we made it more legible and uploaded it to an online repository. This own code [will also be in an online repository when we clean it up.]

This code makes use of the variable values from our second example scenario in (3.7.4)

$$
\begin{aligned}
\eta &= 0.9 \\
\rho &= 0.005 \\
r_1 &= 0.06 \\
\gamma_1 &= 0.03 \\
\gamma_3 &= 0.01 \\
\lambda_1 &= 0.5 \\
\lambda_3 &= 0.5 \\
\delta_3 &= 0.44
\end{aligned} \tag{124}
$$

To which we add β_3, w_3.

$$
\begin{aligned}
w_3 &= 2000 \\
\beta_3 &= 0.5
\end{aligned} \tag{125}
$$

These factors correspond to each movement participant donating $2000 per year, or 5% of a $40.000 salary, and a team of five people being able to convince 5 other people a year on a 20k budget (and mantaining those they have convinced previously.) Further work could be done in order to determine more accurate and realistic estimates. We also consider initial conditions:

$$x_1(t_0) = \texttt{x\_1\_init} = 10^9 \tag{126}$$

$$x_3(t_0) = \texttt{x\_3\_init} = 10^4 \tag{127}$$

We also consider two parameters, corresponding to our unknown constant k_1: `k1_forward_shooting` and `k1_reverse_shooting`. They determine spending on direct work. Their value is such that decreasing it results in too little spending, and the movement accumulates money which is never spent. Conversely, increasing it results in the movement going bankrupt and acquiring infinite debt. However, its value is inexact, and will be a source of error. In particular, if we run simulations until time t, we don't know that the movement will not go bankrupt at some subsequent time, and hence k_1 requires some guesswork. More specifically, if we select the maximum k_1 such that x_1 is positive at time t, we tend to find that $x_1 \to -\infty$ shortly afterwards.

```
k1_forward_shooting = 2*10^(-5)
k1_reverse_shooting = 2*10^(-5)
```

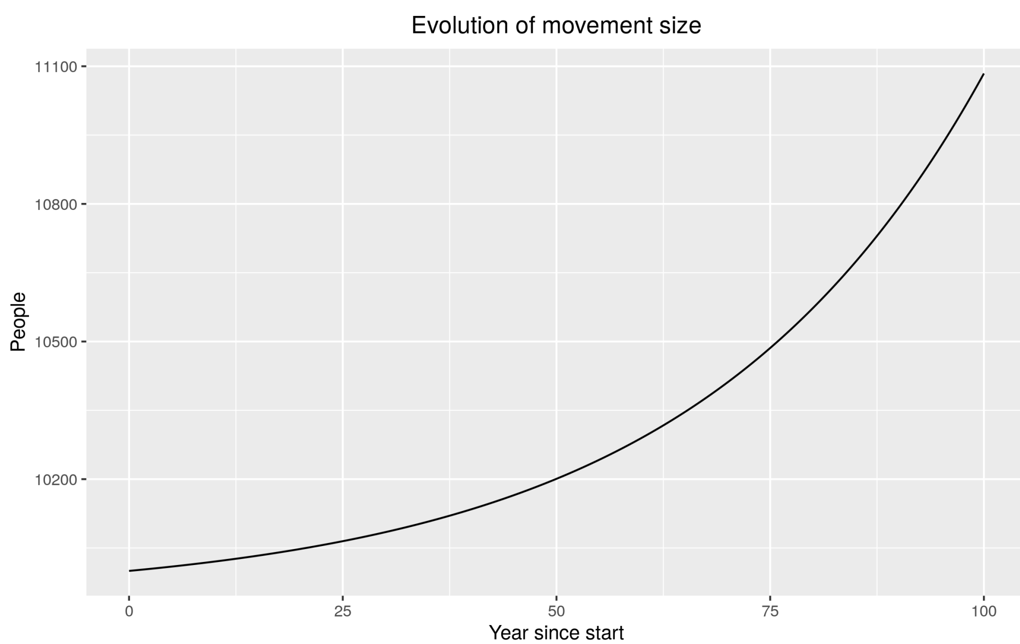
Finally, we decide on a step-size and on a time interval. The time interval will start at 100 years, and increase to 1,000 and then 10,000 years.

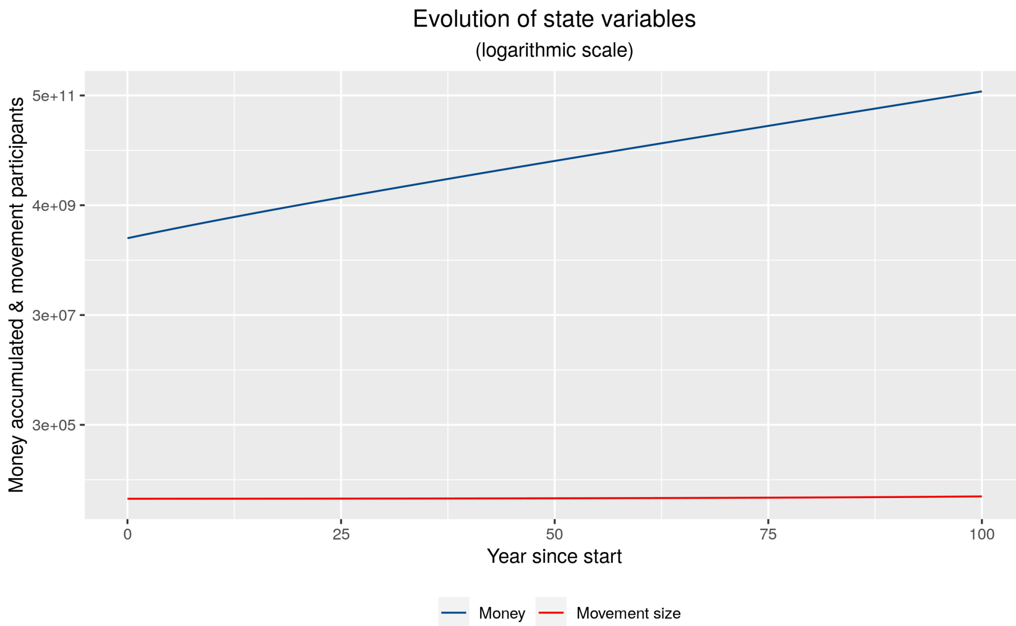
```
stepsize = 0.1 # For now.
first = 0
last = 100
times_forward_shooting = seq(from=first, to=last, by=stepsize)
times_reverse_shooting = seq(from=last, to=first, by=-stepsize)
```

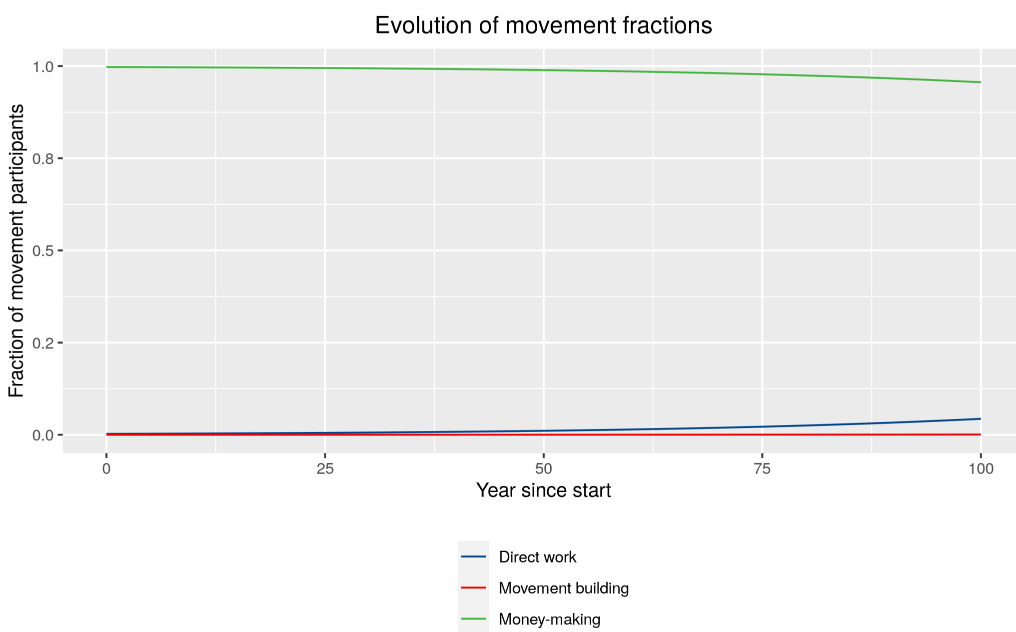
4.2.2 Graphical results: 100 years

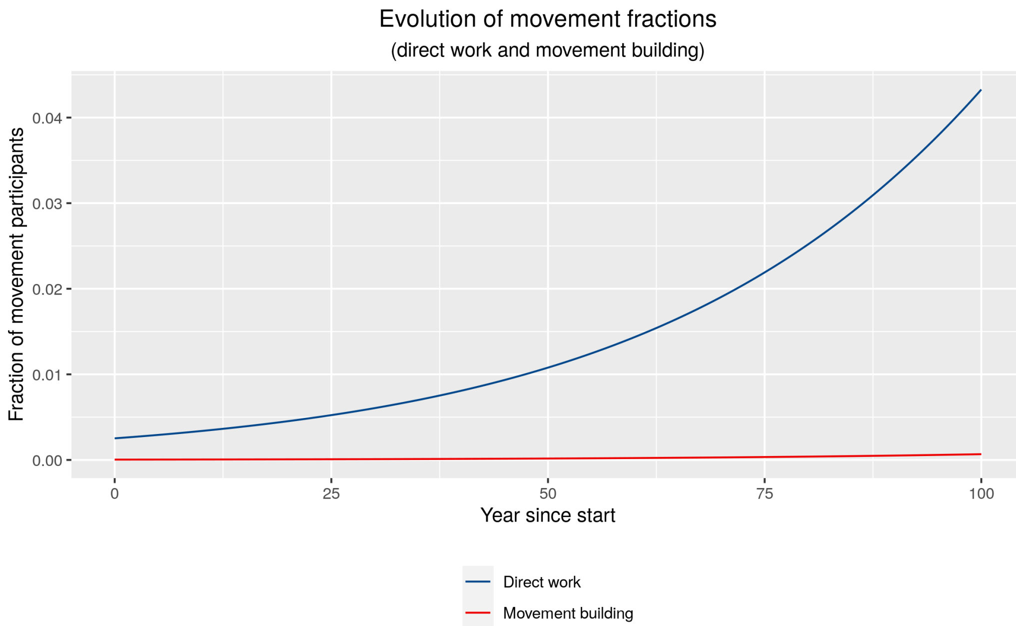
For the first hundred years, accumulated money and movement size grow at different exponential rates. The allocation of participants is primarily to money-making, though both the allocations of movement participants to direct work and to movement building initially increase exponentially, with the former doing so at a much higher rate. Spending also increases in absolute terms for both direct work and movement building (per (4.1)).

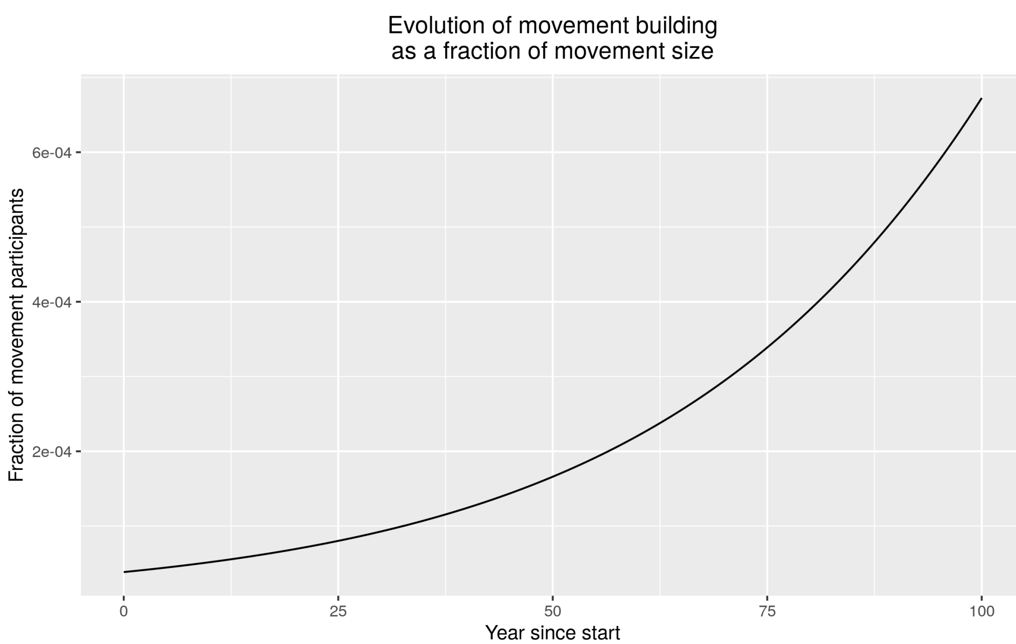

Evolution of state variables
$ & people
6e+11
4e+11
2e+11
0e+00
0
25
50
75
100
Year since start
Money
Movement size

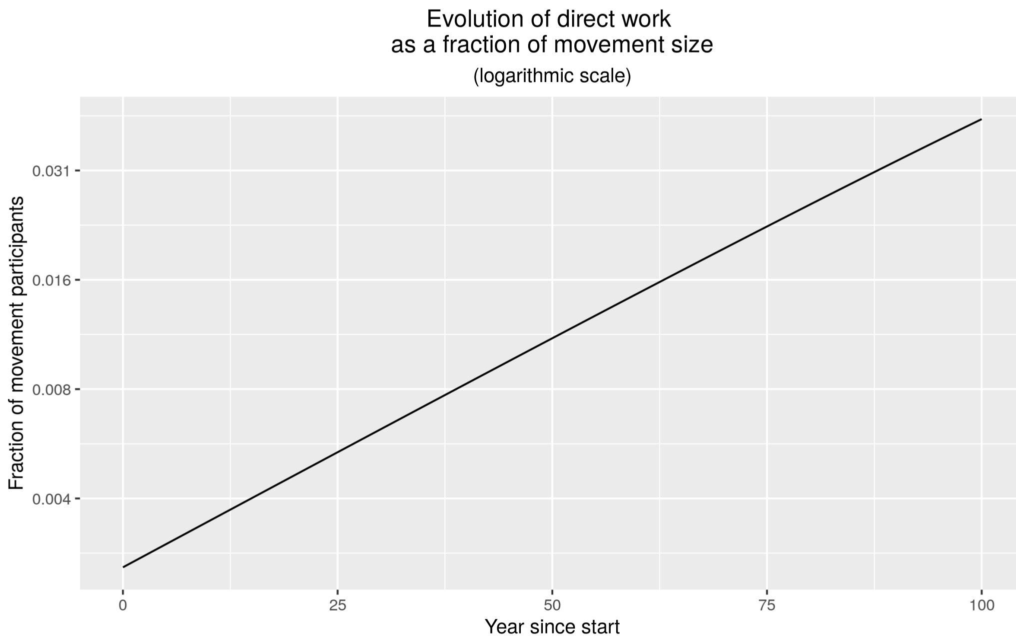

Evolution of movement size
People
11100
10800
10500
10200
0
25
50
75
100
Year since start


Evolution of state variables
(logarithmic scale)
Money accumulated & movement participants
5e+11
4e+09
3e+07
3e+05
0
25
50
75
100
Year since start
Money
Movement size

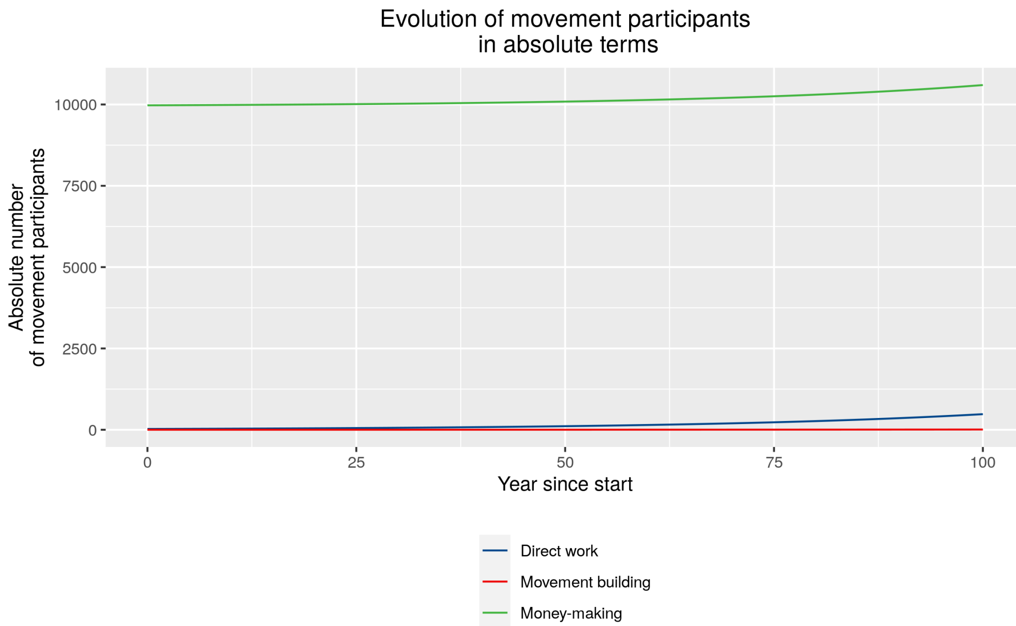

Evolution of movement fractions
Fraction of movement participants
1.0
0.8
0.5
0.2
0.0
0
25
50
75
100
Year since start
Direct work
Movement building
Money-making

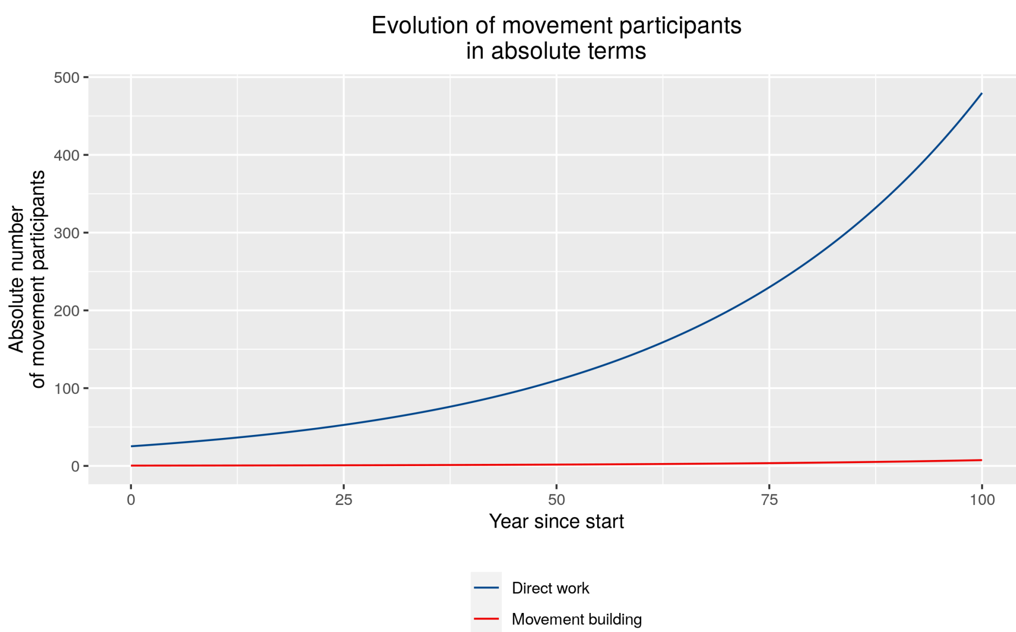

Evolution of movement fractions
(direct work and movement building)
Fraction of movement participants
0.04
0.03
0.02
0.01
0.00
0
25
50
75
100
Year since start
Direct work
Movement building

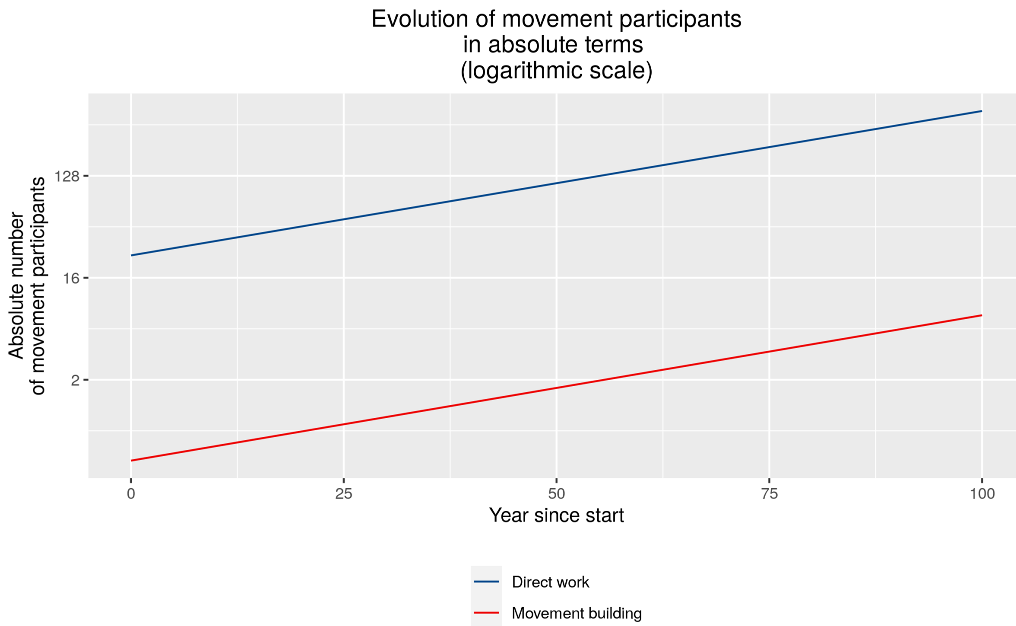

Evolution of movement building
as a fraction of movement size
Fraction of movement participants
6e-04
4e-04
2e-04
0
25
50
75
100
Year since start

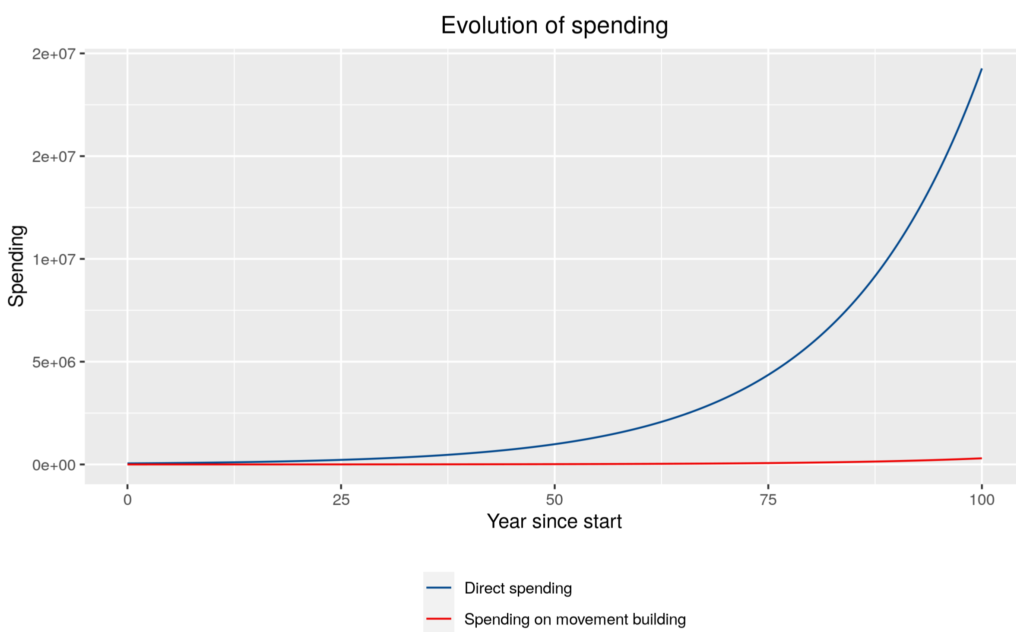

Evolution of direct work
as a fraction of movement size
(logarithmic scale)
Fraction of movement participants
0.031
0.016
0.008
0.004
0
25
50
75
100
Year since start

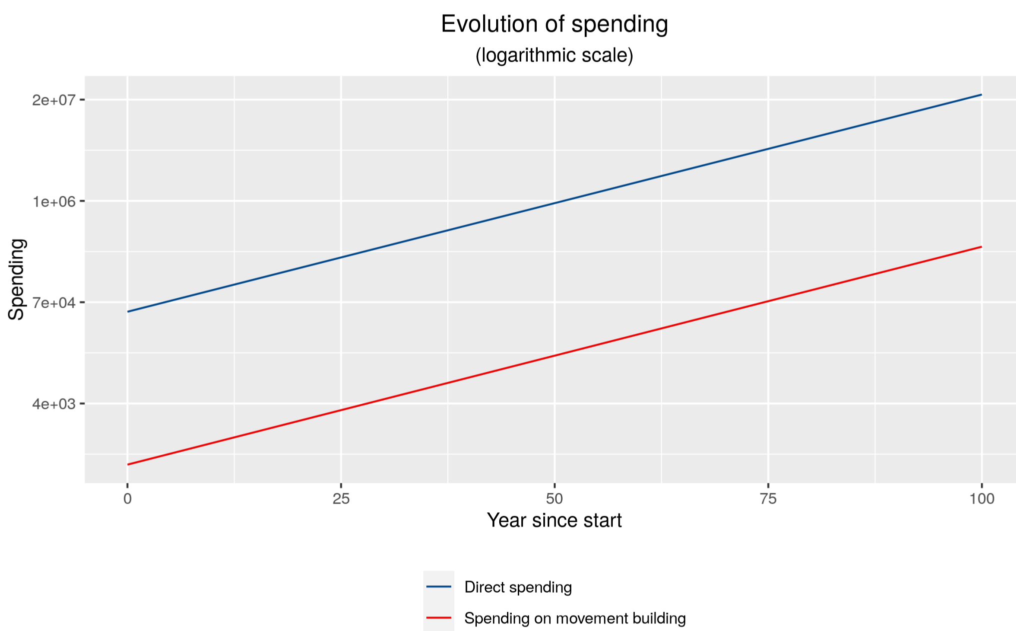

Evolution of movement building
as a fraction of movement size
(logarithmic scale)
Fraction of movement participants
5e-04
2e-04
1e-04
6e-05
0
25
50
75
100
Year since start


Evolution of movement participants
in absolute terms
Absolute number
of movement participants
10000
7500
5000
2500
0
0
25
50
75
100
Year since start
Direct work
Movement building
Money-making


Evolution of movement participants
in absolute terms
Absolute number
of movement participants
500
400
300
200
100
0
0
25
50
75
100
Year since start
Direct work
Movement building

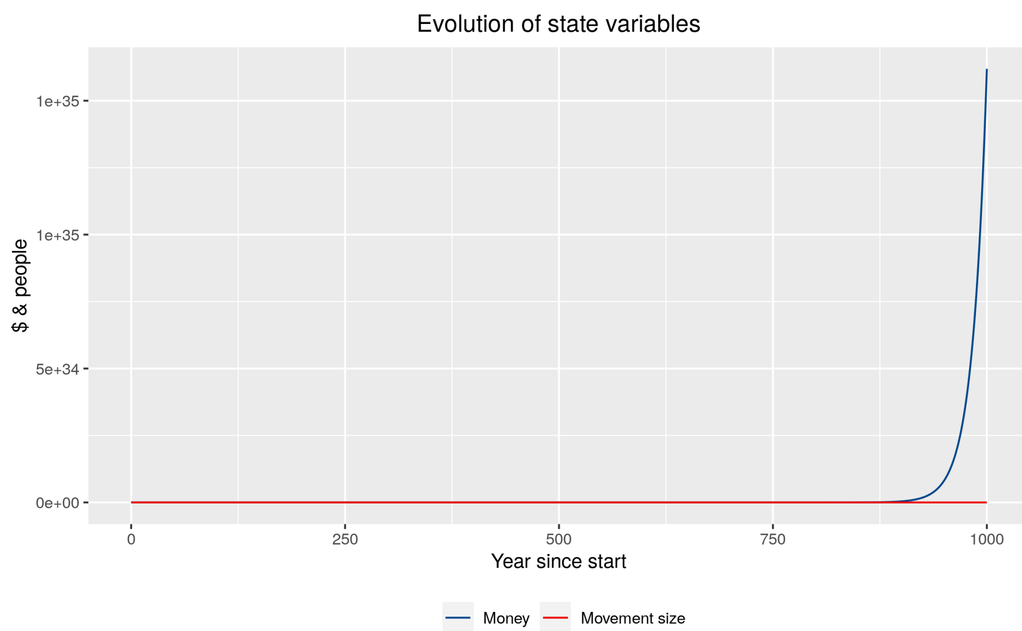

Evolution of movement participants
in absolute terms
(logarithmic scale)
Absolute number
of movement participants
128
16
2
0
25
50
75
100
Year since start
Direct work
Movement building

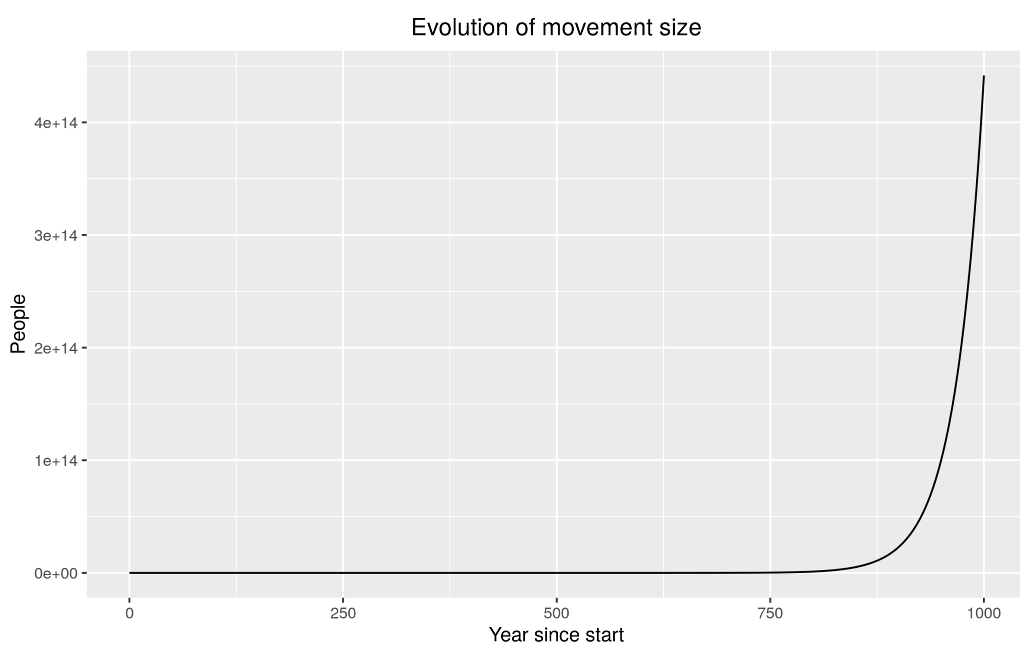

Evolution of spending
2e+07
2e+07
1e+07
5e+06
0e+00
Spending
0
25
50
75
100
Year since start
Direct spending
Spending on movement building

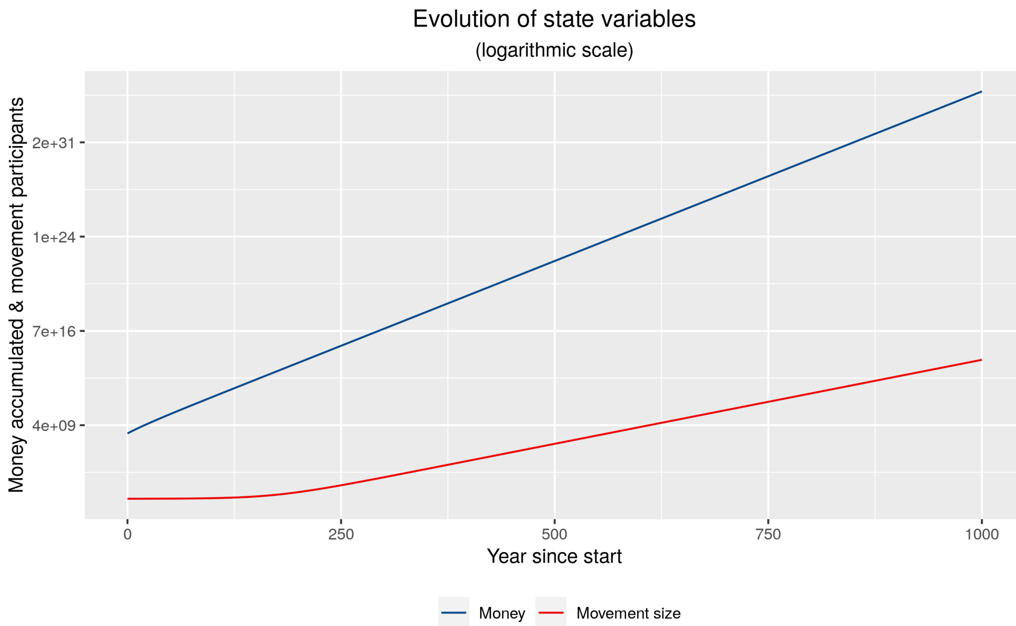


4.2.3 Graphical results: 1,000 years

The dynamic for the state and spending variables is mostly as in the previous section. With regards to movement size and distribution, movement building as a fraction of movement size plateaus at around 0.65%, and stays there. Direct work reaches 40%, and starts slowly declining, whereas money-making starts increasing back-up once again.

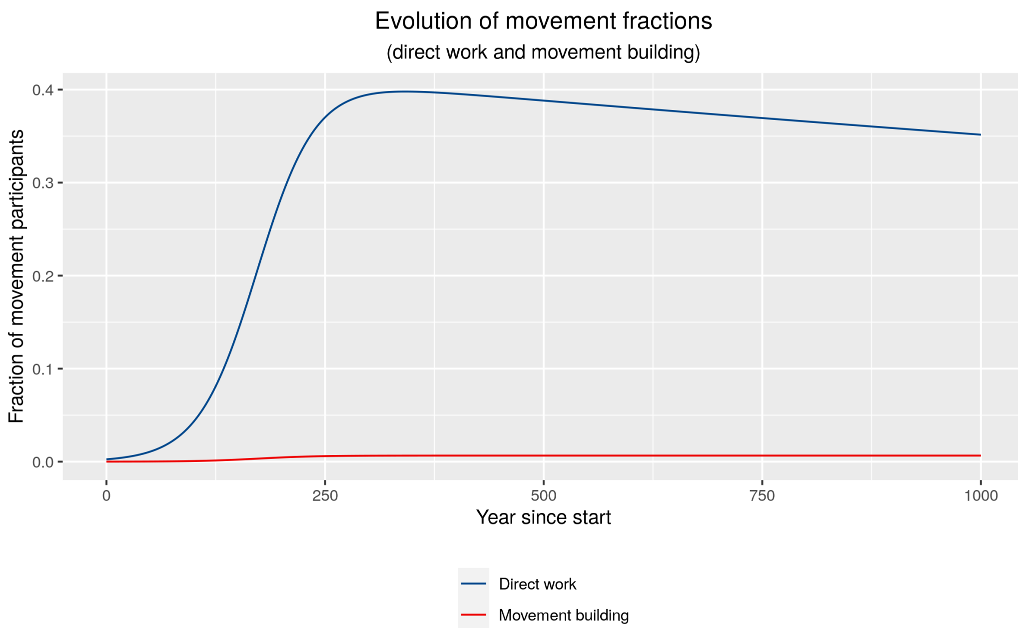

Evolution of state variables
1e+35
1e+35
5e+34
0e+00
$ & people
0
250
500
750
1000
Year since start
Money
Movement size

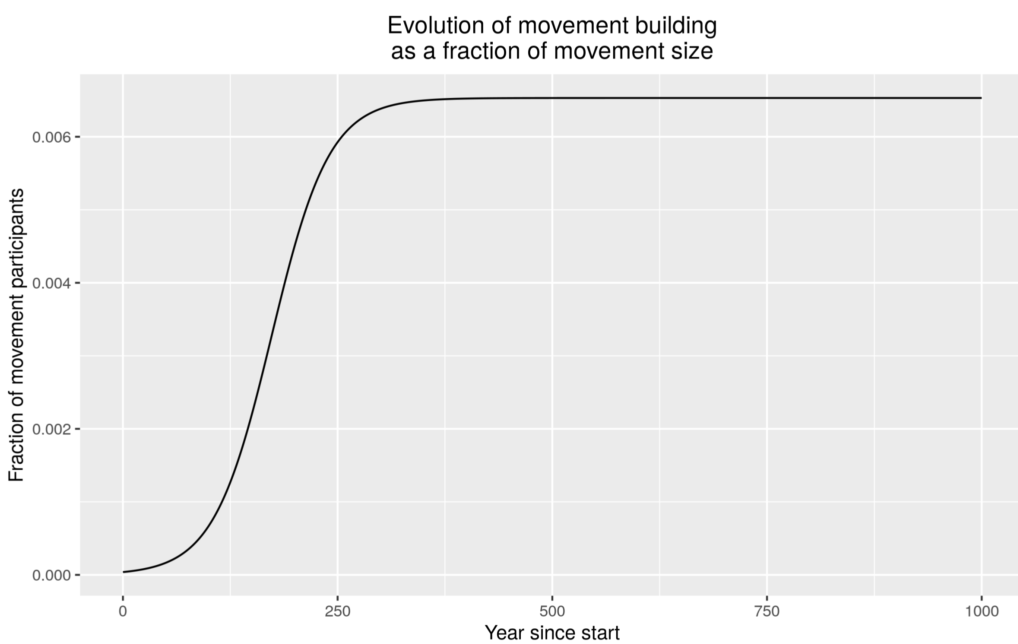

Evolution of movement size
4e+14
3e+14
2e+14
1e+14
0e+00
People
0
250
500
750
1000
Year since start

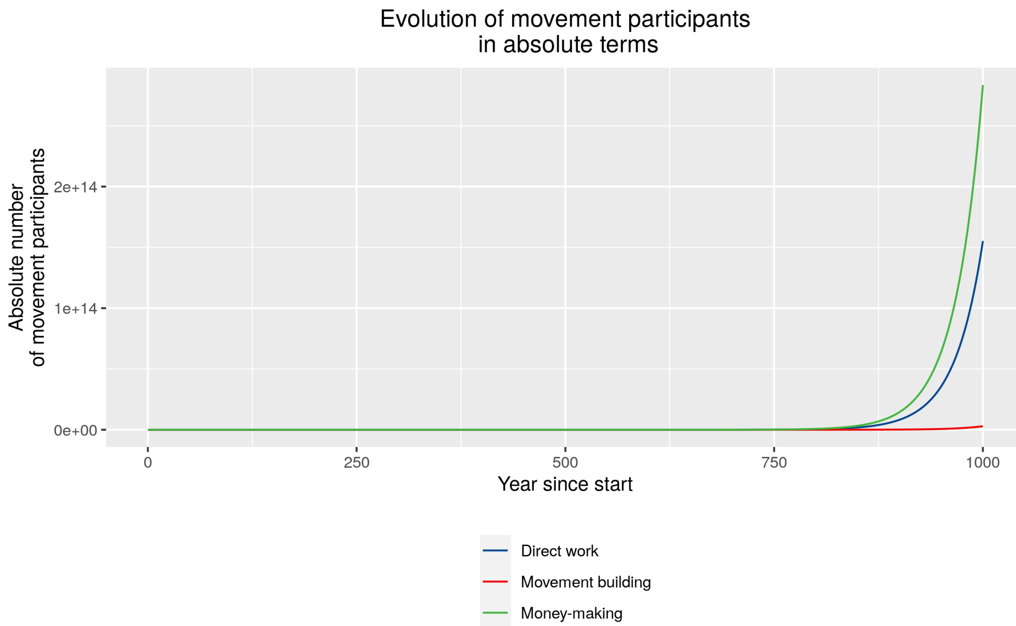


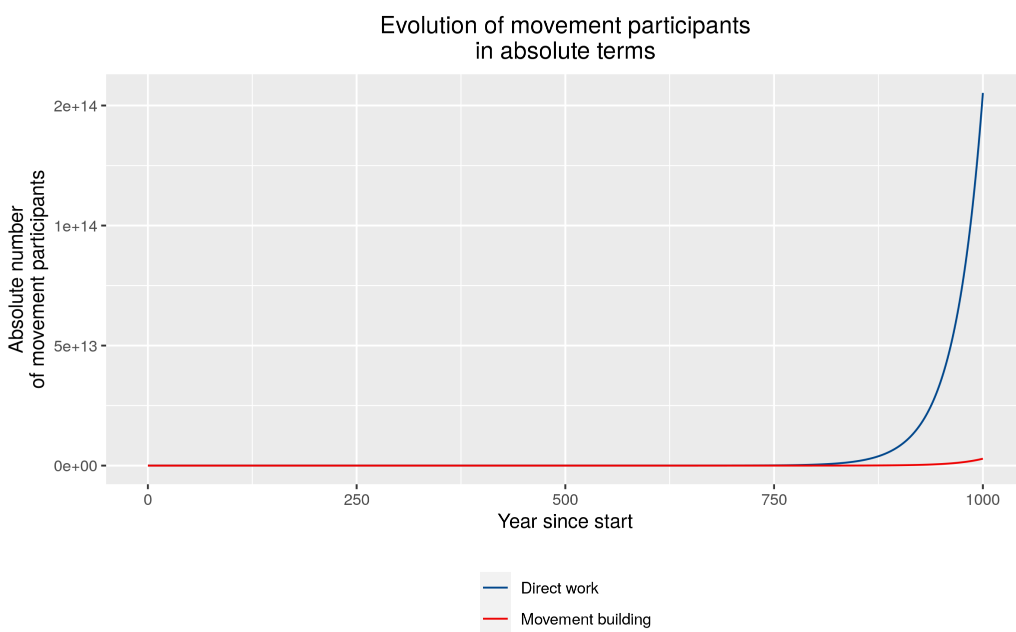
Evolution of state variables
(logarithmic scale)
Money accumulated & movement participants
2e+31
1e+24
7e+16
4e+09
0
250
500
750
1000
Year since start
Money
Movement size

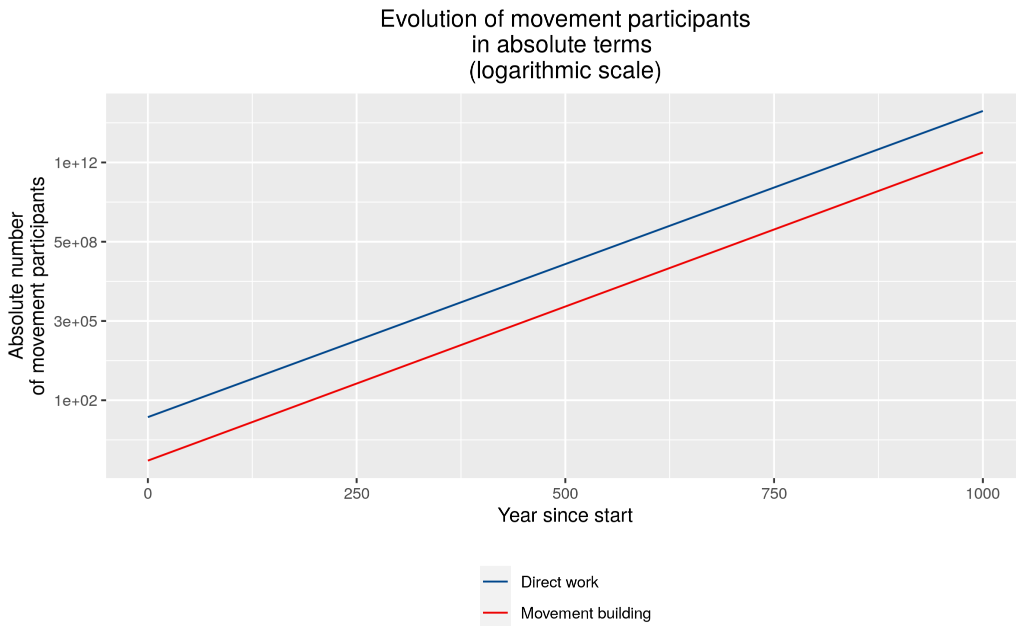
Evolution of movement fractions
Fraction of movement participants
1.0
0.8
0.5
0.2
0.0
0
250
500
750
1000
Year since start
Direct work
Movement building
Money-making

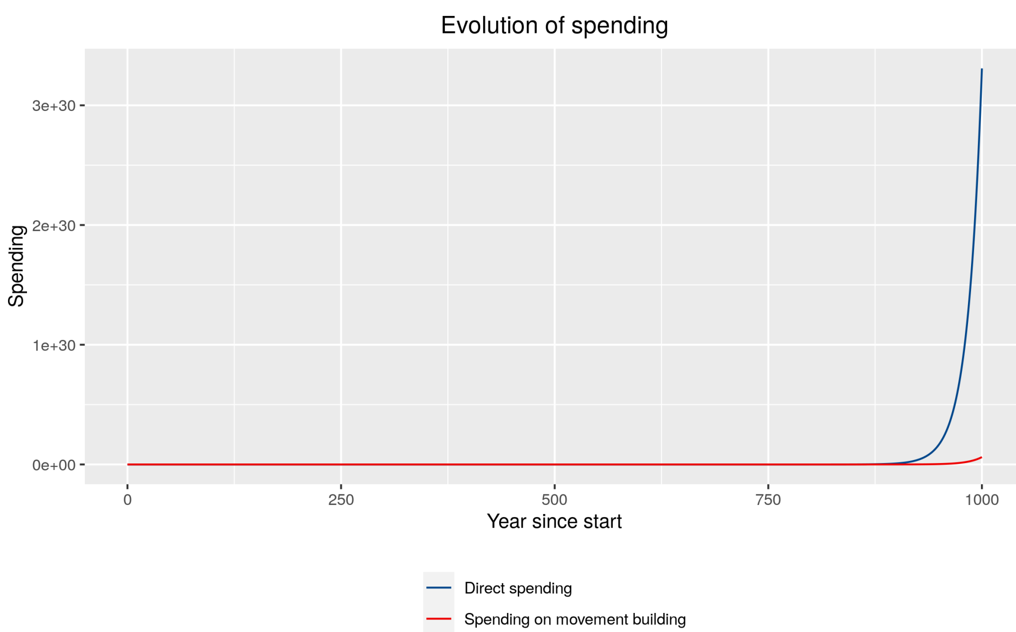

Evolution of movement fractions
(direct work and movement building)
Fraction of movement participants
0.4
0.3
0.2
0.1
0.0
0
250
500
750
1000
Year since start
Direct work
Movement building

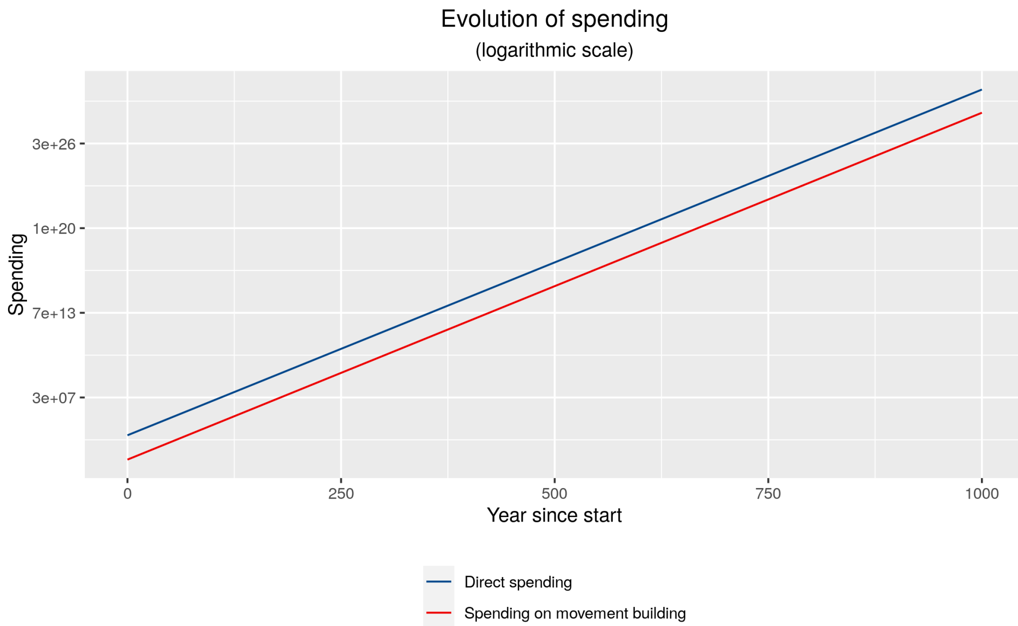

Evolution of movement building
as a fraction of movement size
Fraction of movement participants
0.006
0.004
0.002
0.000
0
250
500
750
1000
Year since start


Evolution of movement participants
in absolute terms
Absolute number
of movement participants
2e+14
1e+14
0e+00
0
250
500
750
1000
Year since start
Direct work
Movement building
Money-making


Evolution of movement participants
in absolute terms
Absolute number
of movement participants
2e+14
1e+14
5e+13
0e+00
0
250
500
750
1000
Year since start
Direct work
Movement building

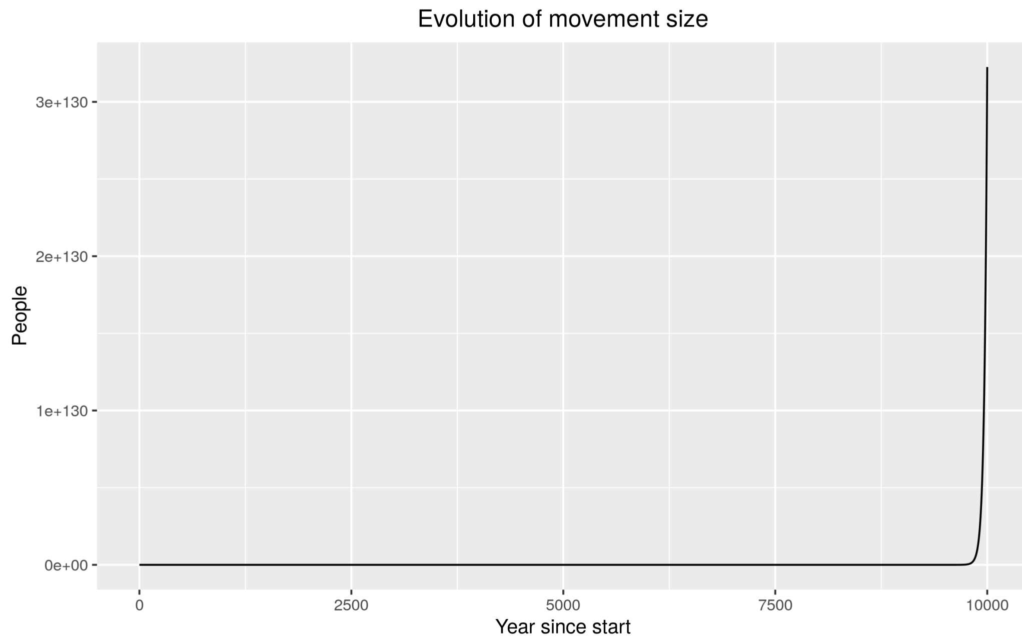

Evolution of movement participants
in absolute terms
(logarithmic scale)
Absolute number
of movement participants
1e+12
5e+08
3e+05
1e+02
0
250
500
750
1000
Year since start
Direct work
Movement building

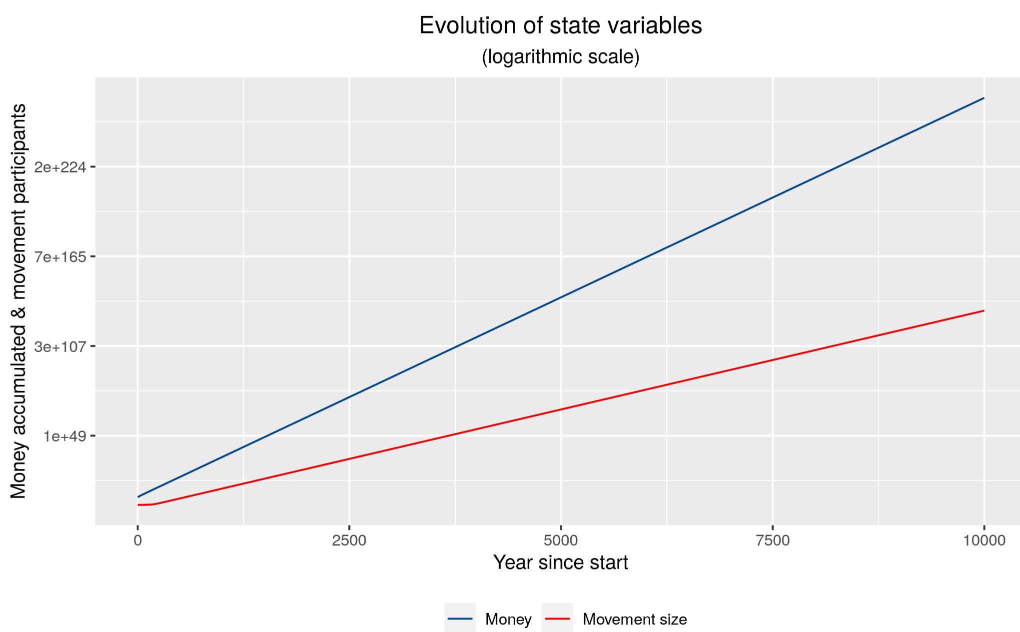

Evolution of spending
Spending
3e+30
2e+30
1e+30
0e+00
0
250
500
750
1000
Year since start
Direct spending
Spending on movement building

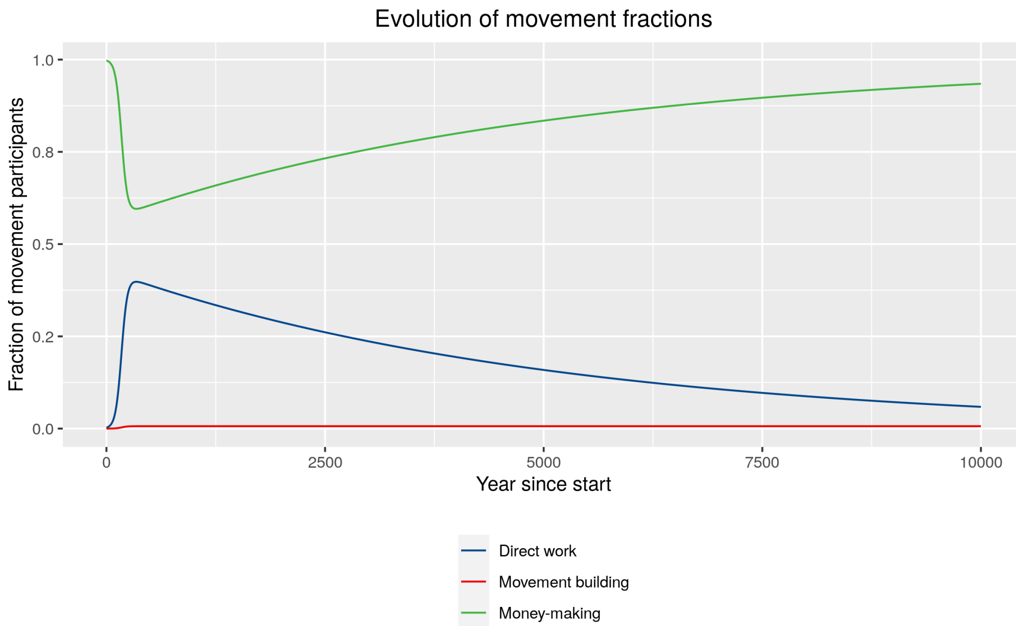


4.2.4 Graphical results: 10,000 years

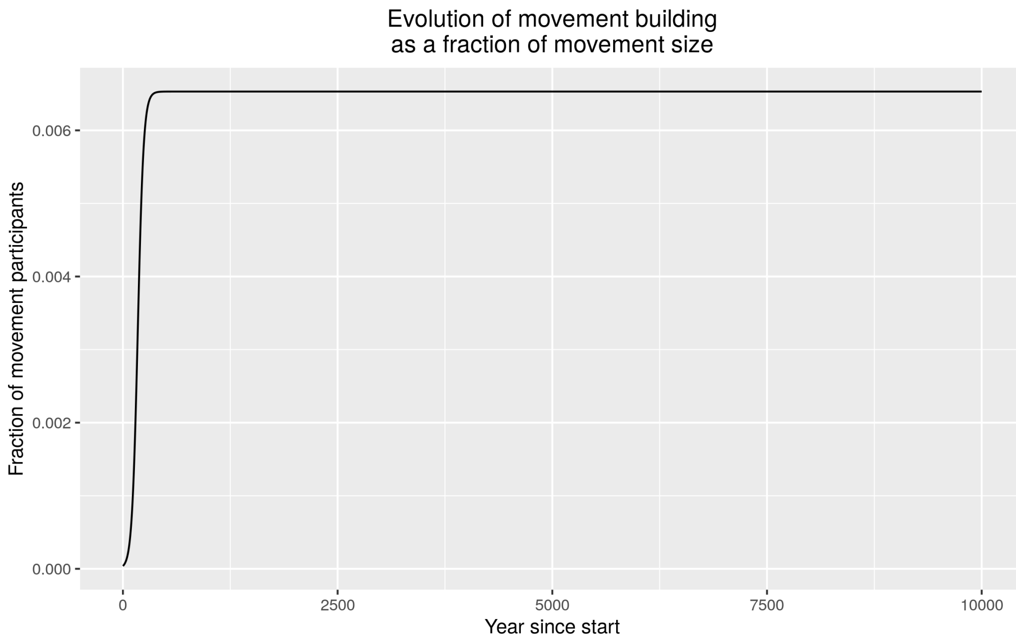
Direct work as a fraction of movement size continues to decrease, perhaps exponentially, but doesn't yet go below movement building. However, we know from the balanced growth rates that it will do so. We can't display some of the graphs on a non-logarithmic scale due to large number limitations in R. [and I'm having some limitations in pushing forward the simulation much beyond 10k years]

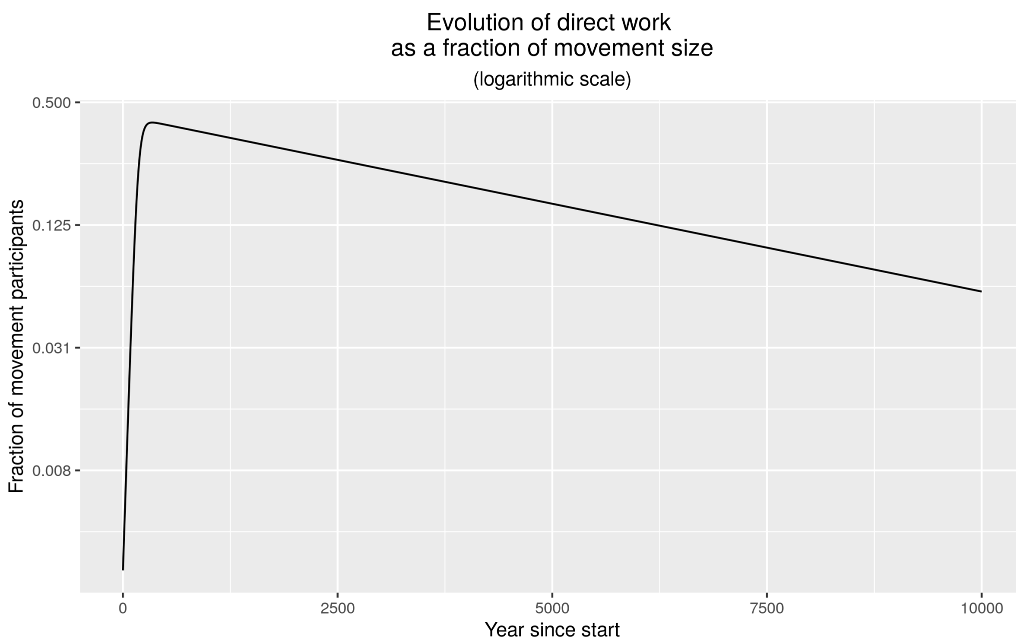

Evolution of movement size
People
3e+130
2e+130
1e+130
0e+00
0
2500
5000
7500
10000
Year since start

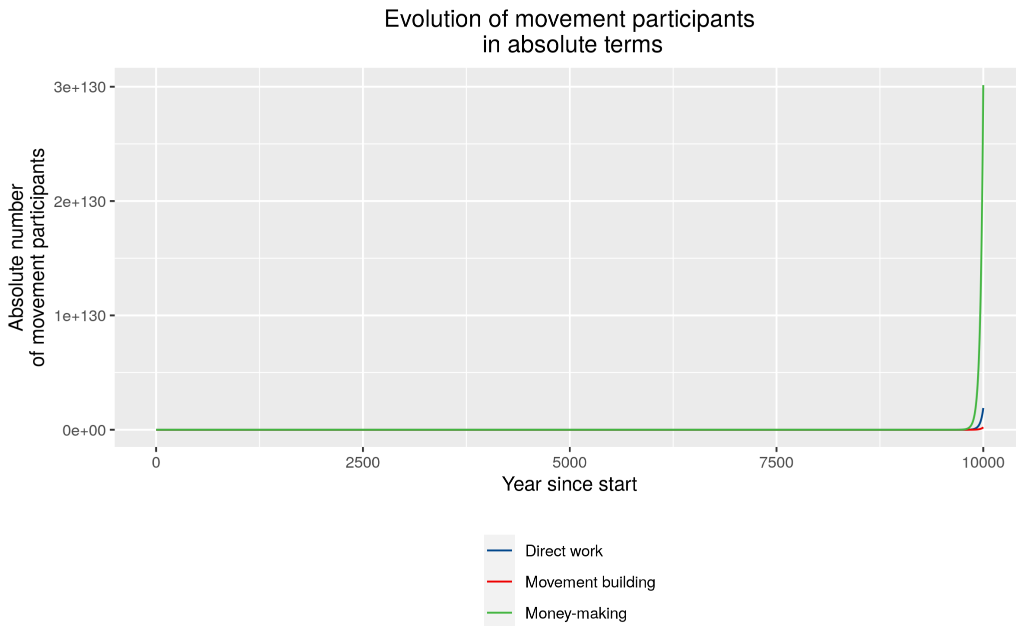

Evolution of state variables
(logarithmic scale)
Money accumulated & movement participants
2e+224
7e+165
3e+107
1e+49
0
2500
5000
7500
10000
Year since start
Money
Movement size

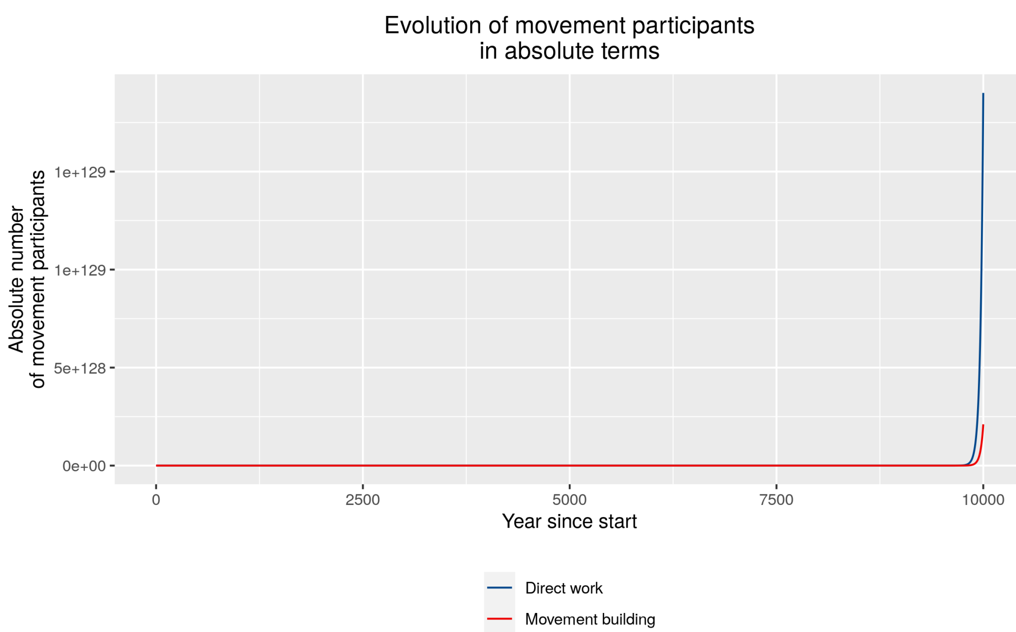

Evolution of movement fractions
Fraction of movement participants
1.0
0.8
0.5
0.2
0.0
0
2500
5000
7500
10000
Year since start
Direct work
Movement building
Money-making

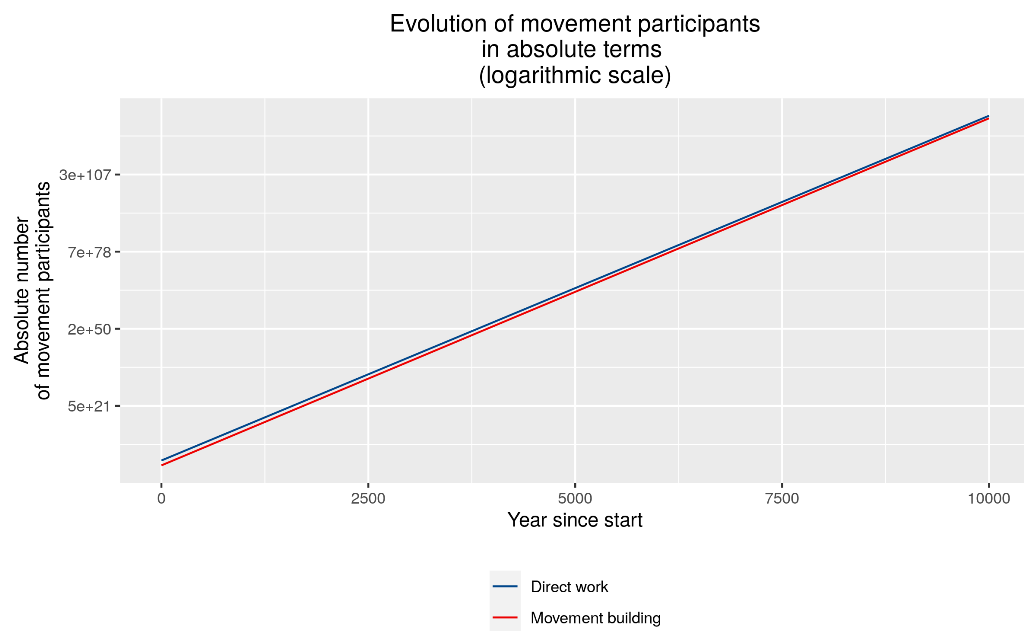

Evolution of movement fractions
(direct work and movement building)
Fraction of movement participants
0.4
0.3
0.2
0.1
0.0
0
2500
5000
7500
10000
Year since start
Direct work
Movement building

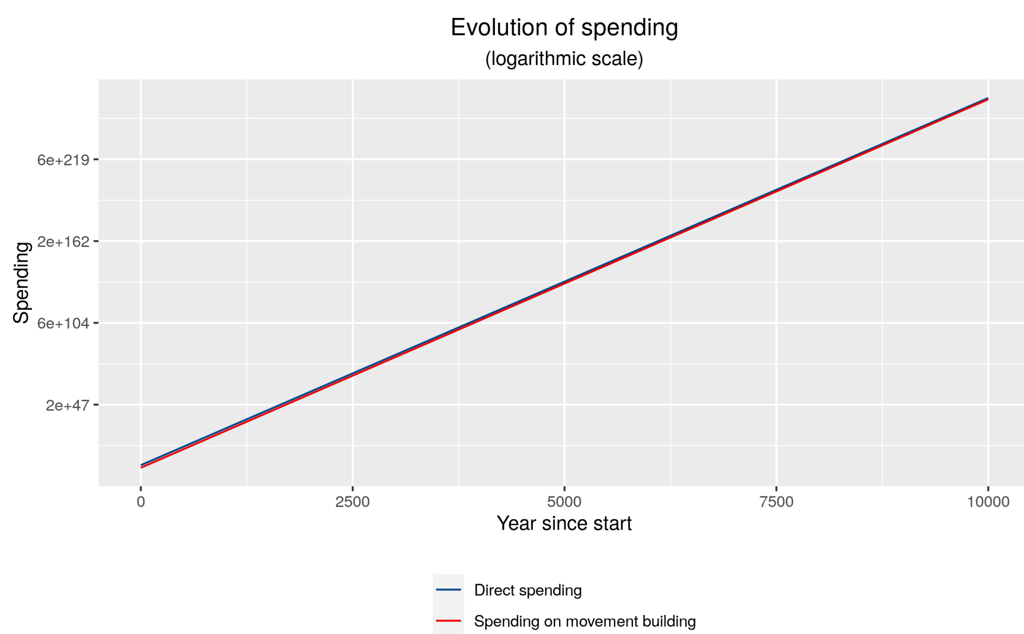

Evolution of movement building
as a fraction of movement size
Fraction of movement participants
0.006
0.004
0.002
0.000
0
2500
5000
7500
10000
Year since start


Evolution of direct work
as a fraction of movement size
(logarithmic scale)
Fraction of movement participants
0.500
0.125
0.031
0.008
0
2500
5000
7500
10000
Year since start


Evolution of movement participants
in absolute terms
Absolute number
of movement participants
3e+130
2e+130
1e+130
0e+00
0
2500
5000
7500
10000
Year since start
Direct work
Movement building
Money-making


Evolution of movement participants
in absolute terms
Absolute number
of movement participants
1e+129
1e+129
5e+128
0e+00
0
2500
5000
7500
10000
Year since start
Direct work
Movement building


Evolution of movement participants
in absolute terms
(logarithmic scale)
Absolute number
of movement participants
3e+107
7e+78
2e+50
5e+21
0
2500
5000
7500
10000
Year since start
Direct work
Movement building


Evolution of spending
(logarithmic scale)
Spending
6e+219
2e+162
6e+104
2e+47
0
2500
5000
7500
10000
Year since start
Direct spending
Spending on movement building

5 Conclusions and discussion

We have considered a stylized model of movement building in the context of social movements which aim to effect some change in the world.

In §3.3 we derived a necessary but not sufficient heuristic which might be used to check whether one is on the optimal path. Its simplest form is, per (41):[2]

$$\frac{\text{\$ spent on direct work}}{\text{\% of people doing direct work}} \cdot \frac{\text{\% of people working on movement building}}{\text{\$ spent on movement building}} = \text{constant}$$

In §3.7, we derived the balanced growth rates for the stylized model, and in §4.1 we derived the spending path, that is, the optimal amount to spend on movement building at any given time.

We also found that, for a space of plausible parameters, the optimal allocation implies an *asymptotic Ponzi* condition, where, even as the number of movement participants doing direct work grows with time in absolute terms, they converge to 0% of the total movement size, with most of the movement participants working either on earning money or in movement building.

However, when carrying out numerical simulations, we find that this asymptotic Ponzi condition is indeed asymptotic, and doesn't instantiate itself in the immediate future.

When carrying out these simulations, we find tend to find that the fraction of movement participants who do direct work grows until it reaches a peak, and then declines with time in favour of the fraction which dedicates themselves to earning money. The exact magnitude and location of this peak depends heavily on the choice of k_1, a difficult to estimate constant, but the overall dynamic of growing and then declining doesn't. [See: appendix or section to discuss this with examples. We can also try to explain this analytically].

The fraction of those who work on movement building seems to grow slowly, until it stabilizes at around 0.5% of total movement size. Regarding movement building, crucially, per (122) we found out that the amount of money spent on movement building is, in a sense, stateless, that is, it doesn't depend on how many movement participants the social movement

[2] To apply this heuristic, find or estimate the four parameters at two distinct points in time. Computing the ratios should result in the same constant at both points in time

already has, but rather solely on their current recruitment costs, their expected contributions, and the rate of return of capital. We provided a graphical illustration of these dynamics in §4.2, for a set of plausible parameters.

We also found that the problem under consideration displays a strong proclivity to violate the transversality conditions, that is, to generate seemingly impossible results. For example, if the amount of money and manpower needed to convince someone to join a social movement is and remains much lower than the amount of money and manpower which typical members are willing to give to this movement, and if these typical members are willing to allocate that money and manpower towards movement building, the optimal solution looks like an almost instantaneous recursive loop which quickly "takes over the world." This is the motivation for the $\delta_3 < 1$ term in (14).

Another fun type of transversality violation are the Satan's apple scenarios, such as those in (Arntzenius et al. 2003) [3]. In these kinds of scenarios, waiting $n + 1$ years might always be strictly better than waiting n years, but waiting forever is strictly worse than waiting any finite amount. In our case, this might correspond to a situation where investing for $n + 1$ years before spending is better than investing for only n years, but where investing forever and never spending is worse than investing for any finite amount of time. Similarly, it might be the case that directing all of a movement's resources and manpower towards movement building for n years to produce explosive movement growth, and then switching over to generating utility is only dominated by doing the same thing for $m > n$ years, but that solely concentrating on movement building forever would be suboptimal.

Now, for a range of plausible parameters this doesn't happen, but there is also no particular reason why one can't fall in a Satan's apple scenario. Arntzenius et al. argue that the rational choice in such a scenario is to stick to a large finite integer and to stop at that point.

Overall, the results above are contingent on the stylized movement building model capturing enough facets of reality to be of interest, but there are many respects in which it is not exhaustive. To mention some salient omissions, we don't consider global catastrophic or existential risks (such as runaway climate change, unaligned artificial intelligence, nuclear brinksmanship, extremely deadly global pandemics, etc.), which might lead us to consider more impatient allocations, and we also don't here consider the interplay between philanthropists who have different rates of time discounting. Further, movement participants are assumed to be immortal. Should it then the case that the stylized model is too far removed from reality, it may still serve as a building blocks for later and more detailed models which take into account these and further considerations.

6 References

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Appendices

A Minimal logarithmic model

A.1 Setup

For the minimal model, returns are logarithmic in the product of research and direct spending. The instantaneous rate of change for research is simply proportional to spending on research. And at each instant, the previous total budget changes compounds by the interest rate, absent the money spent in either direct work or in research. We ignore movement building.

We are maximizing:

$$V(\vec{\alpha(t)}) = \max_{\vec{\alpha(t)}} \int_0^\infty e^{-\rho t} \cdot U(\vec{x(t)}, \vec{\alpha(t)})dt \tag{128}$$

For utility and laws of motion:

$$U(x, \alpha) = ln(x_2 \cdot \alpha_1) \tag{129}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} r_1 x_1 - \alpha_1 - \alpha_2 \\ \alpha_2 \end{bmatrix} \tag{130}$$

under the constraints that

$$\lim_{t \to \infty} x_i \geq 0 \wedge x_2 \geq 0 \wedge \alpha_i \geq 0 \tag{131}$$

With the Hamiltonian standing at:

$$H = ln(x_2 \cdot \alpha_1) + \mu_1 \cdot (r_1 x_1 - \alpha_1 - \alpha_2) + \mu_2 \cdot \alpha_2 \tag{132}$$

and the transversality condition:

$$\lim_{t \to \infty} exp\{-\rho \cdot t\} \cdot x_i \cdot \mu_i = 0 \tag{133}$$

A.2 Optimal path derivation

From constraint (7) on the Hamiltonian:

$$\frac{\partial H}{\partial \alpha_1} = \frac{1}{\alpha_1} - \mu_1 = 0 \implies \alpha_1 = \frac{1}{\mu_1} \tag{134}$$

$$\frac{\partial H}{\partial \alpha_2} = -\mu_1 + \mu_2 = 0 \implies \mu_1 = \mu_2 \tag{135}$$

From constraint (8):

$$-\frac{\partial H}{\partial x_1} = -r_1\mu_1 = \dot{\mu}_1 - \rho\mu_1 \implies \mu_1 = k_1 \cdot exp\{(\rho - r_1)t\} \tag{136}$$

$$-\frac{\partial H}{\partial x_2} = -\frac{1}{x_2} = \dot{\mu}_2 - \rho\mu_2 \tag{137}$$

From (134), (136) we can find the explicit form of α_1:

$$\alpha_1 = \frac{1}{k_1} \cdot exp\{(r_1 - \rho)t\} \tag{138}$$

From (135), (136), (137), we can find the explicit form of x_2:

$$x_2 = \frac{1}{\rho\mu_2 - \dot{\mu}_2} = \frac{1}{k_1 r_1} exp\{(r_1 - \rho)t\} \tag{139}$$

and from (130),(139), a_2:

$$a_2 = \dot{x}_2 = \frac{(r_1 - \rho)}{k_1 r_1} exp\{(r_1 - \rho)t\} = \frac{(1 - \frac{\rho}{r_1})}{k_1} exp\{(r_1 - \rho)t\} \tag{140}$$

Having a_1 and a_2, we can determine x_1 using (130)

$$\dot{x}_1 = r_1 x_1 - \frac{1}{k_1} \cdot exp\{(r_1 - \rho)t\} - \frac{(1 - \frac{\rho}{r_1})}{k_1} exp\{(r_1 - \rho)t\} \tag{141}$$

$$\dot{x}_1 = r_1 x_1 - \frac{2 - \frac{\rho}{r_1}}{k_1} \cdot exp\{(r_1 - \rho)t\} \tag{142}$$

$$x_1 = k_2 \cdot exp\{r_1 t\} + \frac{2 - \frac{\rho}{r_1}}{k_1 \rho} \cdot exp\{(r_1 - \rho)t\} \tag{143}$$

With k_1, k_2 integration constants chosen so that $\vec{x}(0) = \vec{x_0}$. In particular, note that $x_1 < x_{01} \cdot exp\{r_1 t\}$.

A.3 Checking the transversality condition

Consider k_2, the integration constant from solving

$$\dot{x}_1 = r_1 x_1 - \frac{2 - \frac{\rho}{r_1}}{k_1} \cdot exp\{(r_1 - \rho)t\} \tag{144}$$

$$x_1 = k_2 \cdot exp\{r_1 t\} + \frac{2 - \frac{\rho}{r_1}}{k_1 \rho} \cdot exp\{(r_1 - \rho)t\} \tag{145}$$

This constant is uniquely determined by the initial conditions. However, if $k_2 > 0$, then at some point we'll start amassing a fortune which we'll never spend. This is because the k_2 term grows at a rate of r_1, but we're only spending at a rate of $(r_1 - \rho)$. Conversely, if $k_2 < 0$, then at some point we'll go into ever deeper debt, which we never plan to repay.

It is then no coincidence that if $k_2 \neq 0$, the transversality condition (133) doesn't hold, and hence we know the Hamiltonian to output an spurious solution.

$$\begin{aligned}
&\lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot x_1 \cdot \mu_1 \\
&= \lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot \left(k_2 \cdot exp\{r_1 t\} + \frac{2 - \frac{\rho}{r_1}}{k_1 \rho} \cdot exp\{(r_1 - \rho)t\}\right) \\
&\cdot \Big(k_1 \cdot exp\{(\rho - r_1)t\}\Big) \\
&= \begin{cases} 0 \text{ if } k_2 = 0 \\ k_2 \text{ if } k_2 \neq 0, \text{ i.e., the transversality condition doesn't hold} \end{cases}
\end{aligned} \tag{146}$$

We can also check the transversality condition for x_2 and μ_2:

$$\begin{aligned}
&\lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot x_2 \cdot \mu_2 \\
&= \lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot \left(\frac{1}{k_1 r_1} exp\{(r_1 - \rho)t\}\right) \cdot \Big(k_1 \cdot exp\{(\rho - r_1)t\}\Big) \\
&= \lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot \frac{k_1}{k_1 \cdot r_1} = 0
\end{aligned} \tag{147}$$

And this holds, with the possible exception of $k_1 = 0$, i.e., when we start with a fortune of \$0.

A.4 Results and interpretation

When the integration constant k_2 is equal to 0:

$$\alpha_1 = \frac{1}{k_1} \cdot exp\{(r_1 - \rho)t\} \tag{148}$$

$$a_2 = \frac{(1 - \frac{\rho}{r_1})}{k_1} exp\{(r_1 - \rho)t\} \tag{149}$$

$$x_1 = \cancel{k_2 \cdot exp\{r_1 t\}} + \frac{2 - \frac{\rho}{r_1}}{k_1 \rho} \cdot exp\{(r_1 - \rho)t\} \tag{150}$$

$$x_2 = \frac{1}{k_1 r_1} exp\{(r_1 - \rho)t\} \tag{151}$$

In this simple model, both investment in research and direct altruistic spending would grow at the same exponential rate $(r_1 - \rho)$, as long as the rate of returns outpaces the hazard rate ρ.

The shape of research spending differs from that of direct altruistic spending by a multiplicative factor of $(1 - \frac{\rho}{r_1})$, meaning that as the return rate approximates the hazard rate, one would stop spending much on research, though not on direct giving. Commonly, however, $\rho << r_1$, so this multiplicative factor will be close to one.

In particular, for $r = 0.07 = 7\%, \rho = 0.005 = 0.5\%$,

$$\frac{\alpha_1}{\alpha_2} = \frac{1}{1 - \frac{\rho}{r_1}} = \frac{1}{1 - \frac{0.005}{0.07}} = \frac{14}{13} \approx 1.0769 \tag{152}$$

Lastly, total accumulated capital also grows in expectation, although at a lower pace than if it was left to compound alone.

B Some other numerical simulation outputs

[Not yet written.]