Contents

Labor, Capital, and the Optimal Growth of Social Movements

Nuño Sempere

March 31, 2021

WORK IN PROGRESS

1 Introduction

Social movements such as "Effective Altruism" face the problem of optimal allocation of resources across time in order to maximize their desired impact. Much like states and other entities considered in the literature since (Ramsey, 1928) [8], they have the option to invest in order to give more later. However, unlike states, where population dynamics are usually considered exogenous, such agents also have the option of recruiting like-minded associates through movement building. For example, Bill Gates can recruit other ultra-rich people through the Giving Pledge, aspiring effective altruists can likewise spread their ideas, etc.

This paper models the optimal allocation of capital for a social movement between direct spending, investment, and movement building, as well as the optimal allocation of labor between direct workers, money earners, and movement builders. This research direction follows in the footsteps of (Trammell, 2020) [12], which considers the related yet distinct dynamics of a philanthropic funder who aims to provide public goods while having a lower discount rate than less patient partners.

The outline of this paper is as follows: §2 considers a social movement which starts out with a certain amount of capital and a certain number of movement participants. This movement must then decide where to allocate their capital and labor. In §3, we present some theoretical results. In particular, we determine the shape of the optimal path, and remark upon some of its most salient properties. §4 explores some example numerical values for the different coefficients in our model, and we compare the optimal path to a rule of thumb allocation. These example values are then explored more thoroughly in §5, which presents the results from numerical simulations. §6 concludes and outlines implications.

2 Movement Building Model

2.1 Model definition

We seek to maximize:

$$V(\vec{\alpha(t)}) = \max_{\vec{\alpha(t)}} \int_0^\infty e^{-\delta t} \cdot U(L(t), \vec{k}(t), \vec{l}(t))dt \tag{1}$$

For utility and laws of motion:

$$U(t) = U(L(t), \vec{k}(t), \vec{l}(t)) = \frac{(q \cdot k_1(t)^\rho + (1-q) \cdot (L(t) \cdot l_1(t))^\rho)^{\frac{(1-\eta)}{\rho}}}{(1-\eta)} \tag{2}$$

$$\begin{bmatrix} \dot{K}(t) \\ \dot{L}(t) \end{bmatrix} = \begin{bmatrix} r_1 \cdot K(t) - k_1(t) - k_2(t) + L(t) \cdot w_2 \cdot exp\{\gamma_1 \cdot t\} \cdot (1 - l_1(t) - l_2(t)) \\ r_2 \cdot L(t) + \beta_2 \cdot exp\{\gamma_2 t\} \cdot (k_2(t)^{\lambda_2} \cdot (l_2(t) \cdot L(t))^{1-\lambda_2})^{\delta_2} \end{bmatrix} \tag{3}$$

under the constraints that

$$L(t) \geq 0 \wedge k_i(t) \geq 0 \wedge l_1(t) + l_2(t) + l_3(t) = 1 \wedge l_i(t) \geq 0, i = 1, 2 \tag{4}$$

For the following variable definitions:

1. δ, the discount rate per year, either intrinsic discounting (i.e., because we intrinsically care less about the future) or discounting corresponding to the probability of expropriation per year.

2. ρ, the substitution parameter for the constant elasticity production function. As $\rho \to \infty$, inputs become more substitutable, as $\rho \to -\infty$, inputs become less substitutable. When $\rho = 1$, inputs are perfectly substitutable. $\rho \to 0$ produces in the limit the Cobb–Douglas production function.

3. η, the isoelasticiy parameter for the isoleastic utility function applied to our production function. $\eta = 0$ implies constant returns to scale, $\eta < 0$ implies increasing returns to scale, $\eta > 0$ implies diminishing returns. It can also be thought of as the inverse elasticity of intertemporal substitution.

4. $K(t)$, total capital, and $L(t)$, total movement size (labor). Their respective return rates are r_1, the return rate on capital, and r_2, which will typically be negative and represent a decay rate, due to death, value drift on $L(t)$, etc., but could also be positive due to a positive birth or idea-spreading rate, increased productivity per movement member, etc.

5. $k_1(t)$, spending on direct work on a given instant, and $k_2(t)$, the money spent on movement building on a given instant.

6. $l_1(t), l_2(t), l_3(t)$: the fraction of labor which works respectively on direct work, movement building, and money-making[1]. $l_1(t)+l_2(t)+l_3(t) = 1$, so we will substitute $l_3(t) = 1 - l_1(t) - l_2(t)$ throughout. We will interpret $l_3(t) < 0$ as the converse of money-making, namely hiring.

7. $w_2 \cdot exp\{\gamma_1 \cdot t\}$, the factor by which wages rise with time and economic growth, and $\beta_2 \cdot exp\{\gamma_2 t\}$: the changing difficulty of recruiting movement participants with time. For simplicity, we will consider these growth rates —γ_1 and γ_2— to be exogenous.

8. δ_2: movement building returns to scale

2.2 Solution strategy

We define the Hamiltonian to be:

$$H := U(t) + \mu_1(t) \cdot \dot{K}(t) + \mu_2(t) \cdot \dot{L}(t) \tag{5}$$

$$\begin{aligned} H = & \frac{\left(q \cdot k_1(t)^\rho + (1-q) \cdot (L(t) \cdot l_1(t))^\rho\right)^{\frac{(1-\eta)}{\rho}}}{(1-\eta)} \\ & + \mu_1(t) \cdot (r_1 \cdot K(t) - k_1(t) - k_2(t) + L(t) \cdot w_2 \cdot exp\{\gamma_1 \cdot t\} \cdot (1 - l_1(t) - l_2(t))) \\ & + \mu_2(t) \cdot (r_2 \cdot L(t) + \beta_2 \cdot exp\{\gamma_2 t\} \cdot (k_2(t)^{\lambda_2} \cdot (l_2(t) \cdot L(t))^{1-\lambda_2})^{\delta_2}) \end{aligned} \tag{6}$$

[1] In the "Effective Altruism" movement, this is known as "earning to give". For example, a member might take a well paying programming job and donate a portion of it to various global health and development charities, usually following the recommendations of GiveWell

Per Pontryagin's maximum principle, the optimal path to our optimization problem will be given by the solution to the following system of differential equations.

$$\begin{cases} \dfrac{\partial H}{\partial k_1(t)} = 0 \\ \dfrac{\partial H}{\partial k_2(t)} = 0 \\ \dfrac{\partial H}{\partial l_1(t)} = 0 \\ \dfrac{\partial H}{\partial l_2(t)} = 0 \\ -\dfrac{\partial H}{\partial K(t)} = \dot{\mu_1}(t) - \delta \cdot \mu_1(t) \\ -\dfrac{\partial H}{\partial L(t)} = \dot{\mu_2}(t) - \delta \cdot \mu_2(t) \end{cases} \tag{7}$$

Provided that this solution complies with the transversality conditions:

$$\lim_{t \to \infty} exp\{-\delta \cdot t\} \cdot x_i \cdot \mu_i = 0, i = 1, 2 \tag{8}$$

2.3 Hamiltonian equations

From (7) we derive in §A.1 the following system of equations:

$$\mu_1(t) = q \cdot (1 - \eta) \cdot k_1(t)^{\rho - 1} \cdot A \tag{9}$$

$$\mu_1(t) = \mu_2(t) \cdot \lambda_2 \cdot \delta_2 \cdot \frac{B}{k_2(t)} \tag{10}$$

$$\mu_1(t) = \frac{(1 - q) \cdot (1 - \eta)}{w_2} \cdot \frac{(L(t) \cdot l_1(t))^{\rho - 1}}{exp\{\gamma_1 \cdot t\}} \cdot A \tag{11}$$

$$\mu_1(t) = \frac{\delta_2 \cdot (1 - \lambda_2)}{w_2} \cdot \frac{B}{(L(t) \cdot l_2(t)) \cdot exp\{\gamma_1 \cdot t\}} \cdot \mu_2(t) \tag{12}$$

$$\mu_1(t) = c_1 \cdot exp\{(\delta - r_1) \cdot t\} \tag{13}$$

$$\mu_2(t) = \frac{c_1 \cdot w_2}{\delta - r_2} \cdot exp\{(\delta + \gamma_1 - r_1) \cdot t\} = \frac{w_2}{\delta - r_2} \cdot exp\{\gamma_1 \cdot t\} \cdot \mu_1(t) \tag{14}$$

3 Theoretical results

Let $k_1, k_2, l_1, l_2, K, L, \mu_1, \mu_2$ provide a solution to (7). Let us call these the "candidate optimal path". In this section, we will outline various properties of said candidate path. Having done so, proposition 5.1 outlines in which cases the candidate path is indeed the optimal path.

3.1 Variable ratios heuristic

Theorem 1. *In the candidate optimal path, the following relationship for $l_1(t), l_2(t), k_1(t), k_2(t)$ holds:*

$$\frac{k_1(t)}{l_1(t)} = c_3 \cdot exp\left\{\frac{\gamma_1 \cdot \rho}{1-\rho} \cdot t\right\} \cdot \frac{k_2(t)}{l_2(t)} \tag{15}$$

Proof. From (9) + (11):

$$k_1(t) = \left(\frac{1-q}{q \cdot w_2}\right)^{\frac{1}{\rho-1}} \cdot exp\left\{-\frac{\gamma_1}{\rho-1} \cdot t\right\} \cdot (L(t) \cdot l_1(t)) \tag{16}$$

or, equivalently:

$$L(t) = \left(\frac{q \cdot w_2}{1-q}\right)^{\frac{1}{\rho-1}} \cdot exp\left\{\frac{\gamma_1}{\rho-1} \cdot t\right\} \cdot \frac{k_1(t)}{l_1(t)} \tag{17}$$

From (10) + (12):

$$k_2(t) = \frac{w_2 \cdot \lambda_2}{1-\lambda_2} \cdot exp\{\gamma_1 \cdot t\} \cdot (L(t) \cdot l_2(t)) \tag{18}$$

Or, equivalently:

$$L(t) = \frac{1-\lambda_2}{w_2 \cdot \lambda_2} \cdot exp\{-\gamma_1 \cdot t\} \cdot \frac{k_2(t)}{l_2(t)} \tag{19}$$

From (17) and (19), we derive:

$$\frac{k_1(t)}{l_1(t)} = c_3 \cdot exp\left\{\frac{\gamma_1 \cdot \rho}{1-\rho} \cdot t\right\} \cdot \frac{k_2(t)}{l_2(t)} \tag{20}$$

□

Corollary 1.1. *If we replace our constant elasticity of substitution with a Cobb–Douglas production function in (52), then the following relationship holds:*

$$\frac{k_1(t)}{l_1(t)} = c \cdot \frac{k_2(t)}{l_2(t)} \tag{21}$$

Proof. The Cobb–Douglas production function is the analytic limit of the constant elasticity production function as $\rho \to 0$. Because the functions which define our model are well-behaved (analytic, Riemann integrable, etc.), the proof of Theorem 1 also holds in the limit of $\rho \to 0$. This was further verified by carrying out the calculations in §A.2 for the Cobb–Douglas case, which are available upon request. □

This equation on variable ratios is a necessary but not sufficient heuristic, such that a spending schedule which does not satisfy it cannot be optimal under any constant elasticity of substitution or Cobb–Douglas models. Therefore, given $k_i(t), l_i(t), \gamma_1, \rho$ at one given point in time, the variable ratios heuristic serves as a simple "sanity check" on whether a social movement's investments concur with those recommended by our candidate optimal path. In the absence of knowledge about γ_1, or ρ, which might be difficult to estimate, $k_i(t), l_i(t)$ could be checked at more than one point in time.

3.2 Shape of the candidate optimal path and asymptotic growth rates

This section presents the shape of the optimal paths, and the growth rates for the variables in our model: k_1, k_2, l_1, l_2, L, K. These formulas will depend on two integration constants, which are themselves determined by the initial conditions.

Theorem 2. *Let the model described in §2.1 hold. Then the candidate optimal paths are given by:*

$$k_1(t)^{-\eta} = \frac{c_1 \cdot exp\{(\delta - r_1) \cdot t\}}{q \cdot \left(q + exp\left\{\frac{\gamma_1 \cdot \rho}{1-\rho} \cdot t\right\} \cdot (1-q) \cdot \left(\frac{q \cdot w_2}{1-q}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{(1-\eta)}{\rho} - 1}} \tag{22}$$

$$k_2(t) = \left(\frac{w_2 \cdot \lambda_2 \cdot \delta_2}{\delta - r_2} \cdot \left(\frac{1-\lambda_2}{\lambda_2}\right)^{\delta_2 \cdot (1-\lambda_2)}\right)^{\frac{1}{1-\delta_2}} \cdot exp\left\{\left(\frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2} + \gamma_1\right) \cdot t\right\} \tag{23}$$

$$L(t) = c_4 \cdot exp\{r_2 \cdot t\} + \frac{c_3}{g_1 - r_2} \cdot exp\{g_1 \cdot t\} \tag{24}$$

$$l_1(t) = \left(\frac{q \cdot w_2}{1-q}\right)^{\frac{1}{\rho-1}} \cdot exp\left\{\frac{\gamma_1}{\rho - 1} \cdot t\right\} \cdot \frac{k_1(t)}{L(t)} \tag{25}$$

$$l_2(t) = \frac{1-\lambda_2}{w_2 \cdot \lambda_2} \cdot \frac{k_2(t)}{exp\{\gamma_1 \cdot t\} \cdot L(t)} \tag{26}$$

c_1 and c_4 are integration constants, and c_3 and g_1 are given by:

$$c_3 := \frac{\delta - r_2}{\lambda_2 \cdot \delta_2 \cdot w_2} \cdot \left(\frac{w_2 \cdot \lambda_2 \cdot \delta_2}{\delta - r_2} \cdot \left(\frac{1-\lambda_2}{\lambda_2}\right)^{\delta_2 \cdot (1-\lambda_2)}\right)^{\frac{1}{1-\delta_2}} \tag{27}$$

$$g_1 := g(k_2) - \gamma_1 = \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2} \tag{28}$$

Proof. See §A.2. □

Because c_1 and c_4 are free integration constants, the above equations don't represent one candidate optimal path, but rather a family of candidate optimal paths. Initial movement building size and initial capital then later constrain which unique optimal path a specific social movement can afford without running out of funds.

Corollary 2.1. *Let the model described in §2.1 hold. Then the growth rates along the candidate optimal path are given by:*

$$g(k_1) = \frac{r_1 - \delta}{\eta} + \max\left\{0, \frac{\gamma_1 \cdot (1 - \eta - \rho)}{\eta \cdot (\rho - 1)}\right\} \tag{29}$$

$$g(k_2) = \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2} + \gamma_1 \tag{30}$$

$$g(L(t)) = \max(r_2, g_1) = \max\left\{r_2, \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2}\right\} \tag{31}$$

$$g(l_1) = \frac{\gamma_1}{\rho - 1} + g(k_1) - g(L)$$
$$= \frac{\gamma_1}{\rho - 1} + \frac{r_1 - \delta}{\eta}$$
$$+ \max\left\{0, \frac{\gamma_1 \cdot (1 - \eta - \rho)}{\eta \cdot (\rho - 1)}\right\}$$
$$- \max\left\{r_2, \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2}\right\} \tag{32}$$

$$g(l_2) = 0 \tag{33}$$

And the constant to which $l_2(t)$ converges is given by:

$$l_2(t) \to \frac{(g_1 - r_2) \cdot (1 - \lambda_2)}{\delta - r_2} \cdot \delta_2 \tag{34}$$

Proof. See Theorem 2 and §A.2. □

Note that in a model which only has capital, per (Ramsey, 1928) [8], $g(k_1) = \frac{r_1 - \delta}{\eta}$. So the growth in direct spending of our optimal path candidate looks the same as in the capital only model until a threshold, namely $\frac{\gamma_1 \cdot (1-\eta-\rho)}{\eta \cdot (\rho - 1)} > 0$ is reached.

3.3 Asymptotic Single-Mindedness

Theorem 3. *Let the model described in §2.1 hold. Then in almost all cases, the candidate optimal path is asymptotically "single-minded", that is, it asymptotically devotes 100% of labor to either movement building,* $\frac{l_1(t)}{l_1(t)+l_2(t)} \to 0$*, or to direct work,* $\frac{l_2(t)}{l_1(t)+l_2(t)} \to 0$*. The scenarios where neither is the case are a knife-edge case. The first case occurs when*

$$\frac{\gamma_1}{\rho - 1} + \frac{r_1 - \delta}{\eta} + \max\left\{0, \frac{\gamma_1 \cdot (1 - \eta - \rho)}{\eta \cdot (\rho - 1)}\right\} < \max\left\{r_2, \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2}\right) \tag{35}$$

the second case occurs when

$$\frac{\gamma_1}{\rho - 1} + \frac{r_1 - \delta}{\eta} + \max\left\{0, \frac{\gamma_1 \cdot (1 - \eta - \rho)}{\eta \cdot (\rho - 1)}\right\} > \max\left\{r_2, \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2}\right) \tag{36}$$

and the third case, the knife edge, occurs when

$$\frac{\gamma_1}{\rho-1} + \frac{r_1-\delta}{\eta} + \max\left\{0, \frac{\gamma_1 \cdot (1-\eta-\rho)}{\eta \cdot (\rho-1)}\right\} = \max\left\{r_2, \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2}\right) \tag{37}$$

Proof. Per (33), $l_2(t)$ converges to a constant in the optimal path. Per (25) and (32), $l_1(t)$ displays exponential growth, given by:

$$\begin{aligned} g(l_1) = & \frac{\gamma_1}{\rho-1} + \frac{r_1-\delta}{\eta} \\ & + \max\left\{0, \frac{\gamma_1 \cdot (1-\eta-\rho)}{\eta \cdot (\rho-1)}\right\} \\ & - \max\left\{r_2, \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2}\right\} \end{aligned} \tag{38}$$

If $\frac{\gamma_1}{\rho-1} + \frac{r_1-\delta}{\eta} + \max\left\{0, \frac{\gamma_1 \cdot (1-\eta-\rho)}{\eta \cdot (\rho-1)}\right\} - \max\left\{r_2, \frac{\gamma_2+\gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2}\right) < 0$, then $l_1(t) \to 0$ while $l_2(t)$ remains a (positive) constant, and so $\frac{l_1(t)}{l_1(t)+l_2(t)} \to 0$.

If $\frac{\gamma_1}{\rho-1} + \frac{r_1-\delta}{\eta} + \max\left\{0, \frac{\gamma_1 \cdot (1-\eta-\rho)}{\eta \cdot (\rho-1)}\right\} - \max\left\{r_2, \frac{\gamma_2+\gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2}\right\} > 0$, then $l_1(t) \to \infty$ while $l_2(t)$ remains a (positive) constant, and so $\frac{l_2(t)}{l_1(t)+l_2(t)} \to 0$.

In the knife-edge case where $\frac{\gamma_1}{\rho-1} + \frac{r_1-\delta}{\eta} + \max\left\{0, \frac{\gamma_1 \cdot (1-\eta-\rho)}{\eta \cdot (\rho-1)}\right\}$ $- \max\left\{r_2, \frac{\gamma_2+\gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2}\right\} = 0$, both $l_1(t)$ and $l_2(t)$ converge to constants, and the optimal path is not "single-minded". □

Note that even though the relative fraction of movement participants dedicated to either direct work—$l_1(t)$—or to movement building—$l_2(t)$—converges to zero, this might not be the case for the absolute number—namely $l_1(t) \cdot L(t)$ and $l_2(t) \cdot L(t)$)—because, per (24), $L(t)$ also grows exponentially.

Further, note that if $l_1(t) \to \infty$, then the equality $l_1(t) + l_2(t) \leq 1$ is violated. This has the interpretation that when $l_1(t) + l_2(t) > 1$, the movement acquires more labor by hiring it at a $w_2 \cdot exp\{\gamma_1 \cdot t\}$ rate on the open market, that is, the wage which movement participants would have earnt if they had dedicated themselves to money-making.

Now, consider the case where $\gamma_1 = \gamma_2 = 0$. In this case, the wage growth of movement builders matches their productivity growth ($\gamma_2 = 0$). Then, in the more common case where the social movement is hiring additional external labor instead of asking its movement participants to earn money instead of doing direct work ($l_3(t) < 0$), the wage growth of that additional hired labor also corresponds to its productivity growth ($\gamma_1 = 0$).

Corollary 3.1. *Let the model described in §2.1 hold, and let* $\frac{r_1-\delta}{\eta} > 0$. *Then, in the case where wage growth keeps up with productivity growth, (*$\gamma_1 = \gamma_2 = 0$*), the candidate optimal paths pursue direct work with asymptotic single-mindedness* $\left(\frac{l_2(t)}{l_1(t)+l_2(t)} \to 0\right)$ *if*

$$\frac{r_1-\delta}{\eta} > r_2 \tag{39}$$

and pursue movement building with asymptotic single-mindedness $\left(\frac{l_1(t)}{l_1(t)+l_2(t)} \to 0\right)$ *if*

$$\frac{r_1-\delta}{\eta} < r_2 \tag{40}$$

Proof. Follows straightforwardly from Theorem 3.3

□

This intuitively makes sense: if the compounding returns to movement building are higher than the returns to capital adjusted for returns to scale, then most of the labor will work on movement building. Whereas if the compunding returns to movement building are lower than the adjusted returns to capital, spending on movement building makes much less sense.

[Further, Phil strongly suspects that r2 is negative!!, meaning that value drift/death greatly outweigh gains in productivity+spontaneous expansion. If that's the case, we would be far away from the knife edge, and would basically want to concentrate on direct work]

Corollary 3.2. *Let the model described in §2.1 hold, and let* $\frac{r_1-\delta}{\eta} > 0$. *Then, in the case where wages don't keep up with productivity growth (*$\gamma_1 < 0, \gamma_2 > 0$*)*[2]*, the candidate optimal paths pursue direct work with asymptotic single-*

[2] $\gamma_1 < 0$ but $\gamma_2 > 0$ is not a typo. Per (3), if, $\gamma_1 < 0$, with time it costs less and less to hire one unit of productiveness adjusted external labor. In the second case, $\gamma_2 > 0$, one unit of labor spent on movement building produces more results as time goes on. These two effects are equivalent, even if the signs are different.

mindedness $\left(\frac{l_1(t)}{l_2(t)+l_2(t)} \to 0\right)$ *if*

$$\frac{\gamma_1}{\rho-1} + \frac{r_1-\delta}{\eta} > \max\left\{r_2, \frac{\gamma_2+\gamma_1\cdot\delta_2\cdot\lambda_2}{1-\delta_2}\right\} \tag{41}$$

and pursue movement building with asymptotic single-mindedness $\left(\frac{l_1(t)}{l_1(t)+l_2(t)} \to 0\right)$ *if*

$$\frac{\gamma_1}{\rho-1} + \frac{r_1-\delta}{\eta} < \max\left\{r_2, \frac{\gamma_2+\gamma_1\cdot\delta_2\cdot\lambda_2}{1-\delta_2}\right\} \tag{42}$$

Proof. Also follows straightforwardly from Theorem 3.3 □

Note that, for a standard value of $\rho \leq 1$, more negative γ_1—meaning that wages increase slower than productivity—pushes the optimal path towards asymptotic single-mindedness in direct work, whereas less negative γ_1 moves the optimal path towards more movement building.

Similarly, more positive γ_2, perhaps because wages for movement builders increase slower than their productivity, moves the optimal path towards more movement building.

In the case where the relationship between wage and productivity growth is more esoteric—perhaps because wages rise per the Baumol effect even in the face of constant productivity—is given by the full Theorem 3.3. However, the literature, e.g., (Stansbury and Summers, 2017) [11], finds that the growth in wages is lower than the growth in productivity, and models wage growth as dependent on and caused by productivity growth, so it would be surprising if a social movement fell outside the purview of corollaries 3.1 and 3.2.

3.4 The candidate optimal spending paths are relatively independent of initial conditions

Theorem 4. *Let k_1 be the candidate optimal path for direct altruistic spending. Then $k_1(t)$ only depends on the initial conditions (initial capital and initial movement size) by a multiplicative constant, but its growth rate is independent of initial conditions.*

Proof. Per (48) and (29),

$$k_1(t)^{-\eta} = \frac{c_1 \cdot exp\{(\delta - r_1) \cdot t\}}{q \cdot \left(q + exp\left\{\frac{\gamma_1 \cdot \rho}{1-\rho} \cdot t\right\} \cdot (1-q) \cdot \left(\frac{q \cdot w_2}{1-q}\right)^{\frac{\rho}{1-\rho}}\right)^{\frac{(1-\eta)}{\rho}-1}} \tag{43}$$

$$g(k_1) = \frac{r_1 - \delta}{\eta} + \max\left\{0, \frac{\gamma_1 \cdot (1 - \eta - \rho)}{\eta \cdot (\rho - 1)}\right\} \tag{44}$$

The only relationship to initial conditions is c_1, a constant that can only be so high before the path it defines a movement which becomes insolvent, falls into a debt spiral and violates our transversality conditions. This constant depends both on initial capital and initial movement size, and the higher they are, the higher c_1 can be before our transversality condition is violated.

But otherwise, the growth of $k_1(t)$, given by $g(k_1)$, does not depend on initial movement conditions. □

Theorem 5. *The candidate optimal path of spending on movement building, $k_2(t)$, does not depend on initial conditions.*

Proof. Per (98) and (30):

$$k_2(t) = \left(\frac{w_2 \cdot \lambda_2 \cdot \delta_2}{\delta - r_2} \cdot \left(\frac{1-\lambda_2}{\lambda_2}\right)^{\delta_2 \cdot (1-\lambda_2)}\right)^{\frac{1}{1-\delta_2}} \cdot exp\left\{\left(\frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2} + \gamma_1\right) \cdot t\right\} \tag{45}$$

$$g(k_2) = \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2} + \gamma_1 \tag{46}$$

And we observe that neither the initial constant term nor the exponential term depend on initial conditions. □

To explain this independence from initial conditions we might hypothesize that the model is recommending investing into movement building until the discounted rate of return to movement building is lower than the rate of return to capital. For example, the rate of return to capital might be 6% a year and the rate of return to movement building might start out at 10% a year and decrease with further spending. In that case, the model would recommend investing on movement building until the rate of return to movement building drops to just above 6% a year.

But the amount of capital required to bring down the rate of return to movement building from, say, 10% to 6% a year is unrelated to the amount of capital which the social movement in fact possesses. If the movement has larger amounts of capital, then the model described above holds. Otherwise, to follow the recommendations of this model, the social movement would have to raise debt, and then later repay it once it has grown bigger.[3]

If a social movement couldn't raise debt, then such a social movement will experience a phase of rapid and efficient growth (i.e., above the returns to capital) by spending nearly all or nearly of its capital in movement building; we prove that this is the case in §3.5.

Unfortunately, calculating the discounted rate of return to movement building and comparing it to the rate of return to capital is not straightforward. It depends on which proportion of movement participants will participate in direct work, money-making, or movement building. Thus, the explanation above has the status of a hypothesis.

[But is it only a hypothesis? Intuition pump: In the optimal path, the discounted rate of return to movement building must be equal to the rate of return to capital; otherwise you could move money from one to the other??]

Corollary 5.1. *Not all movements will be able to afford the candidate optimal path.*

Proof. Theorem 5 proves that the candidate optimal path of spending on movement building does not depend on initial conditions. It is easy to choose constants such that k_2 is very large from the start, and in particular many times higher than the wages which each participant can earn per year. Then consider a social movement with small amounts of initial capital and few members, and without the ability to raise debt. That movement will not be able to pay the initial $k_2(0)$. □

[3] Because this is an unusual mechanism, it might not be available to a majority of social movements. But it might be available to some, in particular if "raising debt" is not rigidly conceived. For example, ephemeral movements–such as the "squeeze GameStop shorters" movement started in r/WallStreetBets–might be able to carry out an operation equivalent to raising debt from their own members. Social movements also have tools not available to individual borrowers. For instance, social movements might be able to award status to lenders, or movements perceived as respectable–such as the Amish–might be able to stake their credibility on paying a debt back.

Proposition 5.1. *If the social movement is able to afford the candidate optimal path, and the optimal path satisfies the transversality conditions, the candidate optimal path is indeed the optimal path.*

Proof. Follows from applying Pontryagin's maximum principle. □

3.5 Behavior under state constraints

To fully generalize the results in our model to social movements which are not exceedingly wealthy to start with, and to address corollary 5.1, we must impose the constraint that capital may not go below zero ($K(t) \geq 0$)[4]. If we don't, the system of equations which we pose in (7) does still have solutions concordant with the initial quantities of labor and capital. However, those solutions violate the transversality condition: they recommend that the social movement fall ever further into debt, and never repay that debt. If it were possible, this would indeed be optimal, however, that behavior would be unrealistic.

Theorem 6. *In cases where the social movement cannot afford any candidate optimal path, let us impose a $K \geq 0$ constraint. Then, the optimal path is still as in Th. 2 until the constraint is tight ($K = 0$), at which point it differs.*

Proof. See §A.5 □

For example, k_2 is still given by:

$$k_2(t) = \left(\frac{w_2 \cdot \lambda_2 \cdot \delta_2}{\delta - r_2} \cdot \left(\frac{1 - \lambda_2}{\lambda_2} \right)^{\delta_2 \cdot (1-\lambda_2)} \right)^{\frac{1}{1-\delta_2}} \cdot exp \left\{ \left(\frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2} + \gamma_1 \right) \cdot t \right\} \tag{47}$$

until $K = 0$, and k_1 is still given by

$$k_1(t)^{-\eta} = \frac{c_1 \cdot exp\{(\delta - r_1) \cdot t\}}{q \cdot \left(q + exp\left\{ \frac{\gamma_1 \cdot \rho}{1-\rho} \cdot t \right\} \cdot (1-q) \cdot \left(\frac{q \cdot w_2}{1-q} \right)^{\frac{\rho}{1-\rho}} \right)^{\frac{(1-\eta)}{\rho} - 1}} \tag{48}$$

though in this case we do not yet know what c_1 is, nor are we able to approximate it using simulations.

[4] or much below zero ($K \geq$ debt threshold)

4 Numerical exploration

This section assigns numerical values to the variables in the model, and derives what the growth rates for the flow utility are. These are then compared to two rules of thumb.

From (52),

$$U(L(t), k_1(t), k_2(t)) = \frac{(q \cdot k_1(t)^\rho + (1-q) \cdot (L(t) \cdot l_1(t))^\rho)^{\frac{(1-\eta)}{\rho}}}{(1-\eta)} \tag{49}$$

Now, if $\eta > 1$, the utility term will always be negative, and faster decreases are better, all things equal. Conversely, if $\eta < 1$, the utility term will be positive, and faster growth will also be better.

With regards to ρ, if $\rho > 0$ then utility will grow as:

$$g(U) = (1-\eta) \cdot \max\{g(k_1), g(L) + g(l_1)\} \tag{50}$$

Conversely, if $\rho < 0$, that is, if $\rho = -|\rho|$, utility takes the form

$$U(L(t), k_1(t), k_2(t)) = \frac{\left(\frac{q}{k_1(t)^{|\rho|}} + \frac{(1-q)}{(L(t) \cdot l_1(t))^{|\rho|}} \cdot\right)^{\frac{(1-\eta)}{\rho}}}{(1-\eta)} \tag{51}$$

And the growth of $U(t)$ is given by:

$$g(U) = (1-\eta) \cdot \min\{g(k_1), g(L) + g(l_1)\} \tag{52}$$

4.1 Example: $\eta = 1.1$

$$\begin{cases} \eta = \text{Elasticy of spending} = 1.1 \\ \delta = \text{Hazard rate} = 0.005 = 0.5\% \\ \rho = \text{Substitution parameter from the CES production function} = -0.5 \\ q = \text{Share parameter from the CES production function} = 0.5 \\ r_1 = \text{Returns above inflation} = 0.06 = 6\% \\ r_2 = \text{Movement value drift and death rate} = -0.05 \\ \gamma_1 = \text{Change in participant contributions} = 0.03 = 3\% \\ \gamma_2 = \text{Change in the difficulty of recruiting} = 0.01 = 1\% \\ \lambda_2 = \text{Cobb-Douglas elasticity of movement building} = 0.5 \\ \beta_2 = \text{Constant inversely proportional to difficulty of recruiting} = 1 \\ \delta_2 = \text{Elasticity of movement growth} = 0.44 \end{cases} \tag{53}$$

4.1.1 Resulting growth rates

$$\begin{aligned} g(k_1) &= \frac{r_1 - \delta}{\eta} + \max\left\{0, \frac{\gamma_1 \cdot (1 - \eta - \rho)}{\eta \cdot (\rho - 1)}\right\} \\ &= \frac{0.06 - 0.005}{1.1} + \max\left\{0, \frac{0.5 \cdot (1 - 1.1 - (-0.5))}{1.1 \cdot (-0.5 - 1)}\right\} \\ &= 0.05 \end{aligned} \tag{54}$$

$$g(k_2) = \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2} + \gamma_1 = \frac{0.01 + 0.03 \cdot 0.44 \cdot 0.5}{1 - 0.44} + 0.03 = 0.05964285714 \tag{55}$$

$$g(L) = \max(r_2, g_1) = \max(-0.05, 0.05964285714 - 0.03) = 0.02964285714 \tag{56}$$

$$\begin{aligned} g(l_1) &= \frac{\gamma_1}{\rho - 1} + g(k_1) - g(L) \\ &= \frac{0.03}{-0.5 - 1} + 0.05 - 0.02964285714 \\ &= 0.00035714286 \end{aligned} \tag{57}$$

$$g(l_2) = 0 \tag{58}$$

$$\begin{aligned} g(U) &= (1-\eta) \cdot \min\{g(k_1), g(L) + g(l_1)\} \\ &= (1-1.1) \cdot \min\{0.0427, 0.02964285714 + 0.00035714286\} \\ &= -0.003 \end{aligned} \tag{59}$$

4.1.2 Comparison with a rule of thumb allocation

Take a rule of thumb allocation, where $l_1(t) = l_2(t) = 0.5$, and the movement spends 1% of its capital per year, which then grows at 5% per year (i.e., $g_{k_1(t)} = g_{k_2(t)} = g_{K(t)} = 0.05$).

$$g(k_1) = 0.05 \tag{60}$$

$$g(k_2) = 0.05 \tag{61}$$

$$g(l_1) = 0 \tag{62}$$

$$g(l_2) = 0 \tag{63}$$

$$\begin{aligned} g_{L(t)} &= \frac{\gamma_2 + \delta_2 \cdot \lambda_2 \cdot g(k_2) + \delta_2 \cdot (1-\lambda_2) \cdot g(l_2)}{1 - \delta_2 \cdot (1-\lambda_2)} \\ &= \frac{0.01 + 0.44 \cdot 0.5 \cdot 0.05 + 0.44 \cdot (1-0.5) \cdot 0}{1 - 0.44 \cdot (1-0.5)} \\ &= 0.0269 \end{aligned} \tag{64}$$

$$\begin{aligned} g(U) &= (1-\eta) \cdot \min\{g(k_1), g(L) + g(l_1)\} \\ g(U) &= (1-1.1) \cdot \min\{0.05, 0.0269 + 0\} \\ &= -0.00269 \end{aligned} \tag{65}$$

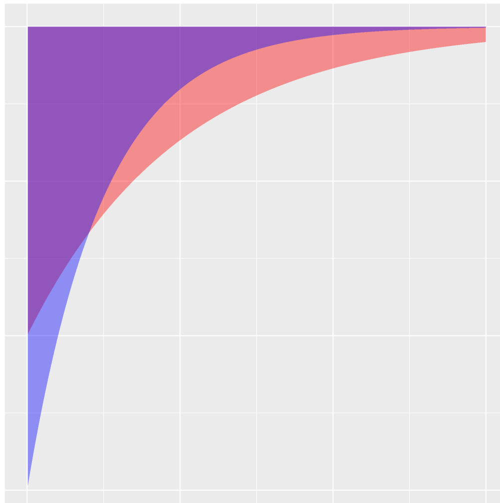


Figure 1: Stylized comparison between optimal allocation (in blue) vs a rule of thumb allocation (in red) in an $\eta > 1$ regime. Not to scale.

4.1.3 Discussion

Note that when $\eta > 1$, as is the case now, the utility term is always negative, and thus all else equal, faster negative growth is preferable. The rule of thumb allocation has a growth of -0.00269, whereas our optimal path has a growth of -0.003, which is as one might expect.

4.2 Example 2: Fastest growing constant fractions path under $\eta = 1.1$

Consider a constant fractions path, i.e., one in which $f_1 := \frac{k_1(t)}{K(t)}$, $f_1 := \frac{k_2(t)}{K(t)}$, $l_1(t)$, $l_2(t)$ are constant, and $l_1 + l_2 = 1$ (i.e., we don't allow for money-making or for hiring). Then, as we derive in A.3, the following are upper bounds for the growth rates for constant fractions paths:

$$g(k_1) \to 0.06 \text{ from below} \tag{66}$$

$$g(k_2) \to 0.06 \text{ from below} \tag{67}$$

$$g(l_1) = 0 \tag{68}$$

$$g(l_2) = 0 \tag{69}$$

$$g(L) \to 0.02974 \text{ from below} \tag{70}$$

$$\begin{aligned} g(U) &= (1-\eta) \cdot \min\{g(k_1), g(L) + g(l_1)\} \\ g(U) &= (1-1.1) \cdot \min\{0.05, 0.02974 + 0\} \\ &= -0.002974 \end{aligned} \tag{71}$$

Note that $g(k_i) \to 0.06$ implies that

$$f_1 \to 0 \text{ from above} \tag{72}$$

$$f_2 \to 0 \text{ from above} \tag{73}$$

i.e., that the social movement should spend a small fraction of its capital at each time period, in order for that capital to grow faster.

So that constant fractions path, which is, in a sense, optimal with respect to the growth rate, has better[5] growth in the utility term than our previous rule of thumb, but worse than the growth rate in the optimal path.

[5]In this case, more negative, because $\eta > 1$

5 Numerical simulations

5.1 Landscape of optimal paths

We can run some simulations to elucidate the short-run behaviour under our candidate optimal path. If we do so, we may land in two regimes. In one case, movement building is hard, discount rates are relatively high, and a small movement such as effective altruism can in fact afford the optimal path. In the other case, movement building is easy, discount rates are low, and a social movement needs to have a large initial endowment to be able to not run out of money before movement building stops yielding high returns, per Proposition 5.1 and Theorem 6. In the following simulations, we consider the former case, and leave the latter case for Appendix C.

We also have to estimate the sign and value of γ_1, γ_2. In Appendix D, we consider the case where the Baumol effect affects the price of direct work labor, but wage grows slower than productivity for movement builders, i.e., when $\gamma_1 > 0, \gamma_2 > 0$. In that case, we also study the behavior around the knife-edge constant from Theorem 3.

In the following sections, we consider a scenario which is far away from the knife edge, and where $\gamma_1 = \gamma_2 = 0$.

[TODO]

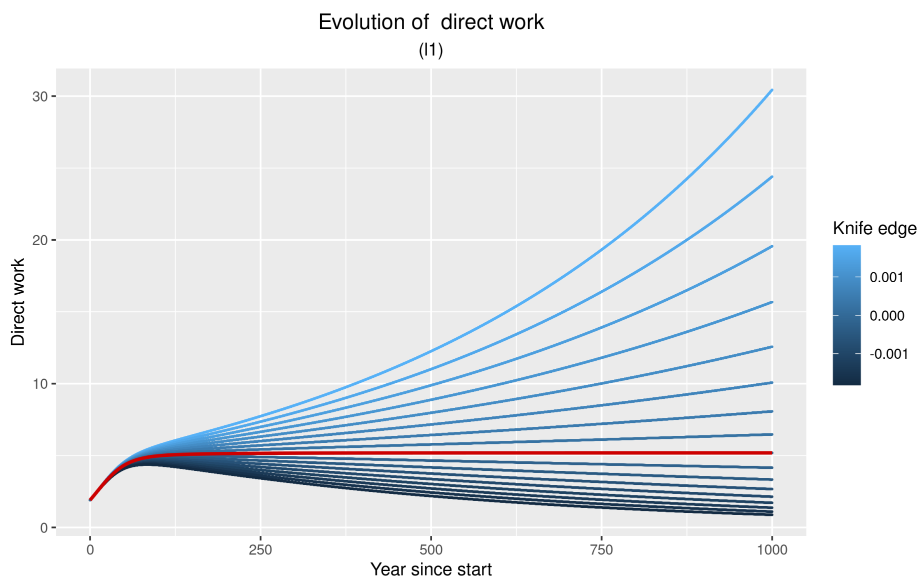
6 Conclusions

[Change with new considerations about wage and productivity]

We have considered a stylized model of movement building in the context of social movements which aim to effect some change in the world. In §2, we presented the details of a movement building model, in §3 we presented some theoretical results, in §4 we explored the numerical values of various growth rates, and in §5 we presented numerical simulation.

We found a candidate optimal path, whose shape and growth rates we provided, and explained in which situations that candidate path truly would be the optimal path. We noticed that the growth of the this path was sometimes like the optimal path in Ramsey's capital-only model, and we proved various propositions about it. Chiefly, it is relatively independent from initial conditions, and it displays "asymptotic single-mindedness".

The asymptotic single-mindedness condition would recommend that, within our candidate optimal path, in almost all cases the social movement ought to direct asymptotically 100% of its labor towards either movement building or towards direct work, depending on an inequality over the variables in the model over what we called a "knife edge constant". Thus, optimal behavior will strongly depend on environment variables which may vary from social movement to social movement. We visualized this dependence when we explored this asymptotic single-mindedness condition numerically, and in particular its relationship to the discount rate:



We also compared the optimal path to various rules of thumb or constrained paths, and showed that the optimal path does provide better growth than optimal rules of thumb.

We noticed that the candidate optimal path, at least with plausibly-sounding parameters, is sometimes ‘too expensive” for relatively small movements, such as Effective Altruism. We hypothesized that this happened because movement building was “too easy”: if movement building or hiring people is scalable and cheap, this leads to a transversality violation. Still, in that case, given our model, a social movement should still follow a path shaped like our candidate optimal path (up to a constant) until capital reaches zero, even if it differs afterwards. In other cases, when movement building is hard enough, there is no transversality violation, and we can safely model and simulate the optimal movement trajectory.

Overall, because of the asymptotic single-mindedness condition, further empirical work to estimate the long-run variable valuables (return to movement building, discount rate, interest rate, etc.) appears to be high value, because it would allow social movements to determine in which side of the asymptotic single-mindedness condition they fall. With regards to Effective Altruism in particular, in some cases our model recommends massive investments into movement building. This would suggest that a ”keep Effective Altruism small/weird” ethos which some in the community hold might not at all be the most effective path.

7 References

[1] Arntzenius, F., Elga, A., & Hawthorne, J. (2004). "Bayesianism, Infinite Decisions, and Binding". Mind, 113(450), new series, pp. 251-283. URL: https://core.ac.uk/download/pdf/11920777.pdf (archived URL). Last accessed March 31, 2021.

[2] Aschenbrenner, L. (2020). "Existential Risk and Growth" (Early draft). URL: https://leopoldaschenbrenner.github.io/xriskandgrowth/ExistentialRiskAndGrowth050.pdf (archived URL). Last accessed March 31, 2021.

[3] Burja, S. (2020). *Long Lived Institutions*. Foresight Institute Talks. URL: https://www.youtube.com/watch?v=6cCcX0xydmk (archived transcript). Last accessed March 31, 2021.

[4] Kurlat, P. (2013). "Continuous Time Optimization, Part 1". URL: https://sites.google.com/view/pkurlat/teaching (archived URL). Last accessed March 31, 2021.

[5] Jones, C. I. (1995). R & D-Based Models of Economic Growth. Journal of Political Economy, 103(4), 759–784.

[6] Jones, C.I. (2016). "Life and Growth". Journal of Political Economy, Vol. 124 (2), pp. 539-578. URL: https://web.stanford.edu/~chadj/LifeandGrowthJPE2016.pdf (archived URL). Last accessed March 31, 2021.

[7] Koopmans, T. C. (1967). Objectives, Constraints, and Outcomes in Optimal Growth Models. Econometrica, 35(1), 1.

[8] Ramsey, F. P. (1928). "A Mathematical Theory of Saving". *The Economic Journal* Vol. 38, No. 152, pp. 543-559. URL: http://piketty.pse.ens.fr/files/Ramsey1928.pdf. (archived URL). Last accessed March 31, 2021.

[9] Romer, P. M. (1986). "Increasing Returns and Long-Run Growth". Journal of Political Economy, 94(5), 1002–1037.

[10] Romer, P. (1986). "Cake Eating, Chattering, and Jumps: Existence Results for Variational Problems". Econometrica, 54(4), pp. 897-908.

URL: https://www.jstor.org/stable/1912842 (archived URL). Last accessed March 31, 2021.

[11] A. M. Stansbury, L. H. Summers (2018) “Productivity and Pay: Is the link broken?” National Bureau of Economic Research, Working Paper URL: https://www.nber.org/papers/w24165

[12] Trammell, P. (2020). “Discounting for Patient Philanthropists” (Early draft). URL: https://philipTrammelll.com/static/discounting_for_patient_philanthropists.pdf (archived URL). Last accessed March 31, 2021.

Appendices

A Proofs and derivations

A.1 Hamiltonian equations

The equations on the Hamiltonian which determine the optimal path are:

$$\frac{\partial H}{\partial k_i} = 0 \tag{74}$$

$$\frac{\partial H}{\partial l_i} = 0 \tag{75}$$

$$-\frac{\partial H}{\partial x_i} = \dot{\mu}_i - \delta \cdot \mu_i \tag{76}$$

For convenience, we also define

$$A := \frac{(q \cdot k_1(t)^\rho + (1-q) \cdot (L(t) \cdot l_1(t))^\rho)^{\frac{(1-\eta)}{\rho}-1}}{(1-\eta)} \tag{77}$$

$$B := \beta_2 \cdot exp\{\gamma_2 t\} \cdot (k_2(t)^{\lambda_2} \cdot (l_2(t) \cdot L(t))^{1-\lambda_2})^{\delta_2} \tag{78}$$

$\frac{\partial H}{\partial k_1(t)} = 0$

$$q \cdot \rho \cdot \frac{(1-\eta)}{\rho} \cdot k_1(t)^{\rho-1} \cdot A - \mu_1(t) = 0 \tag{79}$$

$$\mu_1(t) = q \cdot (1-\eta) \cdot k_1(t)^{\rho-1} \cdot A \tag{80}$$

$\frac{\partial H}{\partial k_2(t)} = 0$

$$-\mu_1(t) + \mu_2(t) \cdot \lambda_2 \cdot \delta_2 \cdot \frac{B}{k_2(t)} = 0 \tag{81}$$

$$\mu_1(t) = \mu_2(t) \cdot \lambda_2 \cdot \delta_2 \cdot \frac{B}{k_2(t)} \tag{82}$$

$\frac{\partial H}{\partial l_1(t)} = 0$

$$(1-q) \cdot L(t)^\rho \cdot l_1(t)^{\rho-1} \cdot \frac{(1-\eta)}{\rho} \cdot \rho \cdot A - L(t) \cdot w_2 \cdot exp\{\gamma_1 \cdot t\} \cdot \mu_1(t) = 0 \tag{83}$$

$$\mu_1(t) = \frac{(1-q)\cdot(1-\eta)}{w_2} \cdot \frac{(L(t)\cdot l_1(t))^{\rho-1}}{exp\{\gamma_1 \cdot t\}} \cdot A \tag{84}$$

$\frac{\partial H}{\partial l_2(t)} = 0$

$$-\mu_1(t)\cdot L(t)\cdot w_2 \cdot exp\{\gamma_1 \cdot t\} + \delta_2 \cdot (1-\lambda_2)\cdot \frac{B}{l_2(t)} \cdot \mu_2(t) \tag{85}$$

$$\mu_1(t) = \frac{\delta_2 \cdot (1-\lambda_2)}{w_2} \cdot \frac{B}{(L(t)\cdot l_2(t))\cdot exp\{\gamma_1 \cdot t\}} \cdot \mu_2(t) \tag{86}$$

$-\frac{\partial H}{\partial K(t)} = \dot{\mu_1}(t) - \delta \cdot \mu_1(t)$

$$-r_1 \cdot \mu_1(t) = \dot{\mu_1}(t) - \delta \cdot \mu_1(t) \tag{87}$$

$$\mu_1(t) = c_1 \cdot exp\{(\delta - r_1)\cdot t\} \tag{88}$$

$-\frac{\partial H}{\partial L(t)} = \dot{\mu_2}(t) - \delta \cdot \mu_2(t)$

$$\begin{aligned} &-\Bigg[(1-q)\cdot\rho\cdot\frac{(1-\eta)}{\rho}\cdot L(t)^{\rho-1}\cdot l_1(t)^{\rho}\cdot A \\ &+\mu_1(t)\cdot w_2\cdot exp\{\gamma_1\cdot t\}\cdot(1-l_1(t)-l_2(t)) \\ &+\mu_2(t)\cdot r_2+\delta_2\cdot(1-\lambda_2)\cdot\frac{B}{L(t)}\Bigg] \\ &=\dot{\mu_2}(t)-\delta\cdot\mu_2(t) \end{aligned} \tag{89}$$

$$-\mu_2(t)\cdot r_2 - \mu_1(t)\cdot w_2 \cdot exp\{\gamma_1 \cdot t\} = \dot{\mu_2}(t) - \delta \cdot \mu_2(t) \tag{90}$$

$$\mu_2(t) = \frac{c_1 \cdot w_2}{\delta - r_2} \cdot exp\{(\delta + \gamma_1 - r_1)\cdot t\} + c_2 \cdot exp\{(\delta - r_2)\cdot t\} \tag{91}$$

Because of an argument about transversality conditions in §A.4, $c_2 = 0$, and hence

$$\mu_2(t) = \frac{c_1 \cdot w_2}{\delta - r_2} \cdot exp\{(\delta + \gamma_1 - r_1)\cdot t\} = \frac{w_2}{\delta - r_2} \cdot exp\{\gamma_1 \cdot t\} \cdot \mu_1(t) \tag{92}$$

A.2 Shape of the optimal path and asymptotic growth rates

A.2.1 For $k_1(t)$

Substituting (17) into (9), we arrive at:

$$\mu_1(t) = q \cdot k_1(t)^{-\eta} \cdot \left(q + exp\left\{ \frac{\gamma_1 \cdot \rho}{1-\rho} \cdot t \right\} \cdot (1-q) \cdot \left(\frac{q \cdot w_2}{1-q} \right)^{\frac{\rho}{1-\rho}} \right)^{\frac{(1-\eta)}{\rho}-1} \tag{93}$$

Or, per (13):

$$k_1(t)^{-\eta} = \frac{c_1 \cdot exp\{(\delta - r_1) \cdot t\}}{q \cdot \left(q + exp\left\{ \frac{\gamma_1 \cdot \rho}{1-\rho} \cdot t \right\} \cdot (1-q) \cdot \left(\frac{q \cdot w_2}{1-q} \right)^{\frac{\rho}{1-\rho}} \right)^{\frac{(1-\eta)}{\rho}-1}} \tag{94}$$

Its growth rate is:

$$g(k_1) = \frac{r_1 - \delta}{\eta} + \max\left\{ 0, \frac{\gamma_1 \cdot (1-\eta-\rho)}{\eta \cdot (\rho - 1)} \right\} \tag{95}$$

The max term arises because, by inspection, if $\frac{\gamma_1 \cdot (1-\eta-\rho)}{\eta \cdot (\rho-1)} < 0$, then the denominator in (94) converges to a constant.

A.2.2 For $k_2(t)$

Substituting (14) into (10), we arrive at:

$$k_2(t) = \frac{w_2 \cdot \lambda_2 \cdot \delta_2}{\delta - r_2} \cdot exp\{\gamma_1 \cdot t\} \cdot B \tag{96}$$

Expanding B:

$$k_2(t) = \frac{w_2 \cdot \lambda_2 \cdot \delta_2}{\delta - r_2} \cdot exp\{\gamma_1 \cdot t\} \cdot \left(k_2(t)^{\lambda_2} \cdot (l_2(t) \cdot L(t))^{1-\lambda_2} \right)^{\delta_2} \cdot \beta_2 \cdot exp\{\gamma_2 \cdot t\} \tag{97}$$

Substituting $(l_2(t) \cdot L(t))$ from (19), and extracting $k_2(t)$, we arrive at:

$$k_2(t) = \left(\frac{w_2 \cdot \lambda_2 \cdot \delta_2}{\delta - r_2} \cdot \left(\frac{1-\lambda_2}{\lambda_2} \right)^{\delta_2 \cdot (1-\lambda_2)} \right)^{\frac{1}{1-\delta_2}} \cdot exp \left\{ \left(\frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2} + \gamma_1 \right) \cdot t \right\} \tag{98}$$

Looking only at the growth rate,

$$g(k_2) = \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2} + \gamma_1 \tag{99}$$

This has the same form as in the Cobb–Douglas case, which isn't surprising since the manipulations we applied were essentially the same. In fact, it doesn't depend on the utility function at all; it's just the rate which maximizes the earning potential of the social movement.

A.2.3 For $L(t)$

Starting from (10), we'll substitute $B = \dot{L}(t) - r_2 \cdot L(t)$.

$$\mu_1(t) = \mu_2(t) \cdot \lambda_2 \cdot \delta_2 \cdot \frac{\dot{L}(t) - r_2 \cdot L(t)}{k_2(t)} \tag{100}$$

$$(\dot{L}(t) - r_2 \cdot L(t)) = \frac{\mu_1(t)}{\mu_2(t) \cdot \lambda_2 \cdot \delta_2} \cdot k_2(t) \tag{101}$$

We'll then substitute $\frac{\mu_1(t)}{\mu_2(t)}$ from (14)

$$\frac{\mu_1(t)}{\mu_2(t)} = \frac{\delta - r_2}{w_2} \cdot exp\{-\gamma_1 \cdot t\} \tag{102}$$

$$(\dot{L}(t) - r_2 \cdot L(t)) = \frac{\delta - r_2}{\lambda_2 \cdot \delta_2 \cdot w_2} \cdot exp\{-\gamma_1 \cdot t\} \cdot k_2(t) \tag{103}$$

and $k_2(t)$ from (98)

$$\begin{aligned}(\dot{L}(t) - r_2 \cdot L(t)) = &\frac{\delta - r_2}{\lambda_2 \cdot \delta_2 \cdot w_2} \cdot exp\{-\gamma_1 \cdot t\} \\ &\cdot \left(\frac{w_2 \cdot \lambda_2 \cdot \delta_2}{\delta - r_2} \cdot \left(\frac{1-\lambda_2}{\lambda_2} \right)^{\delta_2 \cdot (1-\lambda_2)} \right)^{\frac{1}{1-\delta_2}} \\ &\cdot exp \left\{ \left(\frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2} + \gamma_1 \right) \cdot t \right\}\end{aligned} \tag{104}$$

For simplicity,

$$c_3 := \frac{\delta - r_2}{\lambda_2 \cdot \delta_2 \cdot w_2} \cdot \left(\frac{w_2 \cdot \lambda_2 \cdot \delta_2}{\delta - r_2} \cdot \left(\frac{1-\lambda_2}{\lambda_2} \right)^{\delta_2 \cdot (1-\lambda_2)} \right)^{\frac{1}{1-\delta_2}} \tag{105}$$

$$g_1 := g(k_2) - \gamma_1 = \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2} \tag{106}$$

And then

$$(\dot{L}(t) - r_2 \cdot L(t)) = c_3 \cdot exp\{g_1 \cdot t\} \tag{107}$$

$$\dot{L}(t) = c_3 \cdot exp\{g_1 \cdot t\} + r_2 \cdot L(t) \tag{108}$$

So finally:

$$L(t) = c_4 \cdot exp\{r_2 \cdot t\} + \frac{c_3}{g_1 - r_2} \cdot exp\{g_1 \cdot t\} \tag{109}$$

Where c_4 is an integration constant, which could later be determined to be 0, but c_3 is defined in (105)

A.2.4 For $l_1(t)$

$k_1(t)$ and $L(t)$ are known, so per (16) $l_1(t)$ is also known, and given by:

$$l_1(t) = \left(\frac{q \cdot w_2}{1-q} \right)^{\frac{1}{\rho-1}} \cdot exp\left\{ \frac{\gamma_1}{\rho - 1} \cdot t \right\} \cdot \frac{k_1(t)}{L(t)} \tag{110}$$

We can also consider its growth:

$$g(l_1) = \frac{\gamma_1}{\rho - 1} + g(k_1) - g(L) \tag{111}$$

$$g(l_1) = \frac{\gamma_1}{\rho - 1} + \frac{r_1 - \delta}{\eta} + \max\left\{ 0, \frac{\gamma_1 \cdot (1 - \eta - \rho)}{\eta \cdot (\rho - 1)} \right\} - \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2} \tag{112}$$

A.2.5 For $l_2(t)$

$k_2(t)$ and $L(t)$ are known, so per (18) $l_2(t)$ is also known, and given by:

$$l_2(t) = \frac{1-\lambda_2}{w_2 \cdot \lambda_2} \cdot \frac{k_2(t)}{exp\{\gamma_1 \cdot t\} \cdot L(t)} \tag{113}$$

But, looking at (24) $g(L) = \gamma_1 + g(k_2)$ (since either $r_2 < 0$ or $r_1 < g_1$). Hence, $g(l_2) = 0$, that is, $l_2(t)$ converges to a constant. Specifically:

$$l_2(t) \to \frac{(g_1 - r_2) \cdot (1-\lambda_2)}{\delta - r_2} \cdot \delta_2 \tag{114}$$

A.2.6 For $K(t)$

We can also obtain $K(t)$ from its law of motion:

$$\dot{K}(t) = r_1 \cdot K(t) - k_1(t) - k_2(t) + L(t) \cdot w_2 \cdot exp\{\gamma_1 \cdot t\} \cdot (1 - l_1(t) - l_2(t)) \tag{115}$$

Abstracting constants away, and taking into account that $g(l_2) = 0$, this results in

$$\begin{aligned} K(t) = {} & a \cdot exp\{r_1 \cdot t\} - b \cdot exp\{g(k_1) \cdot t\} - c \cdot exp\{g(k_2) \cdot t\} \\ & + d \cdot exp\left\{\big(g(L) + \gamma_1\big) \cdot t\right\} + e \cdot exp\left\{\big(g(L) + \gamma_1 + g(l_1)\big) \cdot t\right\} \end{aligned} \tag{116}$$

A.3 Fastest growing constant fractions path

Let a constant fractions path be a path where l_1, l_2, f_1, f_2 are constants, with $f_1 := \frac{k_1(t)}{K(t)}$, $f_2 := \frac{k_2(t)}{K(t)}$, and with $l_1 + l_2 = 1$. That is, the fractions of capital spent at each period into direct work and into movement building remain constant, and so do the fractions of labor.

Then, per (3),

$$K'(t) = (r_1 - f_1 - f_2) \cdot K(t) \tag{117}$$

and hence

$$K(t) = K_0 \cdot \exp\{(r_1 - f_1 - f_2) \cdot t\} \tag{118}$$

Similarly,

$$
\begin{aligned}
L'(t) = {} & r_2 \cdot L(t) \\
& + f_2 \cdot \beta_2 \cdot l_2^{(1-\lambda_2)\cdot\delta_2} \\
& \cdot exp\{(\gamma_2 \cdot (r_1 - f_1 - f_2) \cdot \lambda_2 \cdot \delta_2) \cdot t\} \cdot L(t)^{(1-\lambda_2)\cdot\delta_2}
\end{aligned} \tag{119}
$$

Which is solved by the following expression

$$
\begin{aligned}
L(t) = \Big(& I_1 \cdot exp\{(1-(1-\lambda_2)\cdot\delta_2)\cdot r_2 \cdot t\} \\
& + \frac{f_2 \cdot \beta_2 \cdot l_2^{(1-\lambda_2)\cdot\delta_2} \cdot (1-(1-\lambda_2)\cdot\delta_2)}{\gamma_2 + (r_1 - f_1 - f_2) - (1-(1-\lambda_2)\cdot\delta_2)\cdot r_2} \\
& \cdot exp\{(\gamma_2 + (r_1 - f_1 - f_2)\cdot\lambda_2\cdot\delta_2)\cdot t\}\Big)^{\frac{1}{1-(1-\lambda)\cdot\delta}}
\end{aligned} \tag{120}
$$

Where I_1 is an integration constant determined by $L(0) = L_0$.

Now, because $(1-(1-\lambda_2)\cdot\delta_2)\cdot r_2$ will generally be smaller than $\gamma_2 + (r_1 - f_1 - f_2)\cdot\lambda_2\cdot\delta_2$, and provided that $f_2 \neq 0$:

$$
g(L) = \frac{\gamma_2 + (r_1 - f_1 - f_2)\cdot\lambda_2\cdot\delta_2}{1-(1-\lambda)\cdot\delta} \tag{121}
$$

Now, the growth in the utility term is given by

$$
g(U) = (1-\eta)\cdot \min\{g(k_1), g(L) + g(l_1)\} \tag{122}
$$

Because $g(k_1) = g(K)$ (when $f_1 \neq 0$), and $g(l_1) = 0$ this reduces to

$$
g(U) = (1-\eta)\cdot \min\left\{(r_1 - f_1 - f_2), \frac{\gamma_2 + (r_1 - f_1 - f_2)\cdot\lambda_2\cdot\delta_2}{1-(1-\lambda)\cdot\delta}\right\} \tag{123}
$$

This growth is generally maximized when $f_1, f_2 = 0$, but they can't be exactly zero, because then k_1 and L would respectively be zero or have a lower growth rate. In any case, $f_1 = f_2 = 0$ provides an upper bound for the growth rate in the utility term of constant fraction paths which, as we see in §4.2, is exceeded by the candidate optimal path.

A.4 Transversality conditions

A.4.1 For $K(t), \mu_1(t)$

The transversality condition reads

$$\lim_{t\to\infty} exp\{-\delta \cdot t\} \cdot K(t) \cdot \mu_1(t) = 0 \tag{124}$$

and we know from (116) and (13) that

$$\begin{aligned} K(t) &= a \cdot exp\{r_1 \cdot t\} - b \cdot exp\{g(k_1) \cdot t\} - c \cdot exp\{g(k_2) \cdot t\} \\ &\quad + d \cdot exp\left\{\big(g(L) + \gamma_1\big) \cdot t\right\} + e \cdot exp\left\{\big(g(L) + \gamma_1 + g(l_1)\big) \cdot t\right\} \end{aligned} \tag{125}$$

$$\mu_1(t) = c_1 \cdot exp\{(\delta - r_1) \cdot t\} \tag{126}$$

Substituting into (124)

$$\begin{aligned} &\lim_{t\to\infty} exp\{-\delta \cdot t\} \cdot c_1 \cdot exp\{(\delta - r_1) \cdot t\} \cdot \\ &\quad \Big(a \cdot exp\{r_1 \cdot t\} - b \cdot exp\{g(k_1) \cdot t\} - c \cdot exp\{g(k_2) \cdot t\} \\ &\quad + d \cdot exp\left\{\big(g(L) + \gamma_1\big) \cdot t\right\} + e \cdot exp\left\{\big(g(L) + \gamma_1 + g(l_1)\big) \cdot t\right\}\Big) = 0 \end{aligned} \tag{127}$$

$$\begin{aligned} &\lim_{t\to\infty} c_1 \cdot exp\{-r_1 \cdot t\} \cdot \Big(a \cdot exp\{r_1 \cdot t\} - b \cdot exp\{g(k_1) \cdot t\} - c \cdot exp\{g(k_2) \cdot t\} \\ &\quad + d \cdot exp\left\{\big(g(L) + \gamma_1\big) \cdot t\right\} + e \cdot exp\left\{\big(g(L) + \gamma_1 + g(l_1)\big) \cdot t\right\}\Big) = 0 \end{aligned} \tag{128}$$

From this, we deduce that $a = 0$, and that the following constraints must hold:

$$\begin{cases} g(k_1) < r_1 \\ g(k_2) < r_1 \\ g(L) + \gamma_1 < r_1 \\ g(L) + \gamma_1 + g(l_1) < r_1 \end{cases} \tag{129}$$

A.4.2 For $L(t), \mu_2(t)$

We want to calculate $l_1(t)$, $l_2(t)$ by substituting $k_i(t), L(t)$ in (16) and (16). For this, we want to check that c_2 and c_4 are indeed equal to 0.

The transversality condition reads

$$\lim_{t\to\infty} exp\{-\delta \cdot t\} \cdot L(t) \cdot \mu_2(t) = 0 \tag{130}$$

We know that:

$$L(t) = c_4 \cdot exp\{r_2 \cdot t\} + \frac{c_3}{g_1 - r_2} \cdot exp\left\{\frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2} \cdot t\right\} \tag{131}$$

$$\mu_2(t) = \frac{c_1 \cdot w_2}{\delta - r_2} \cdot exp\{(\delta + \gamma_1 - r_1) \cdot t\} + c_2 \cdot exp\{(\delta - r_2) \cdot t\} \tag{132}$$

Commonly, $\delta + \gamma_1 - r_1 < 0$, $r_2 < \rho < 0$ and $\frac{\gamma_2+\gamma_1 \cdot \delta_2 \cdot \lambda_2}{1-\delta_2} > 0$

If $c_2 \neq 0$, then $\delta - r_2 > 0$ and $\mu_2(t) \to \infty$ as $t \to \infty$. But $L(t) \to \infty$ as well, so the transversality condition doesn't hold. So $c_2 = 0$.

Now, if $r_2 < 0$, then the $c_4 \cdot exp\{r_2 \cdot t\} \to 0$ as $t \to \infty$, so that term doesn't have that much relevance in the long term. And thus the transversality condition ends up being

$$\lim_{t\to\infty} exp\{-\delta \cdot t\} \cdot exp\{(\delta + \gamma_1 - r_1) \cdot t\} \cdot exp\left\{\frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2} \cdot t\right\} \to 0 \tag{133}$$

Or

$$-\delta + \delta + \gamma_1 - r_1 + \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2} < 0 \tag{134}$$

Which, simplified, ends up being

$$\gamma_1 + \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2} = \gamma_1 + g(L) = g(k_2) < r_1 \tag{135}$$

A.5 Behavior under state constraints

[Note: this section is worded pretty informally, and has to be cleaned up.]

We can pose the problem with a constraint in the state variable. To do this, we first require $K(t) \geq 0$. Then, we incorporate a penalty term $\pi(t)$ to our Hamiltonian such that $\pi(t)$ is only nonzero when $K(t) = 0$, and zero otherwise. This is sometimes expressed as:

$$\pi(t) \cdot K(t) = 0 \tag{136}$$

And we incorporate the penalty term into the Hamiltonian as follows:

$$\hat{H} := H + \pi(t) \cdot K(t) \tag{137}$$

where $\hat{H}$ denotes the Hamiltonian with the penalty term, and H denotes the Hamiltonian without it. Thus,

$$\hat{H} = U(t) + \mu_1(t) \cdot \dot{K}(t) + \mu_2(t) \cdot \dot{L}(t) + \pi(t) \cdot K(t) \tag{138}$$

Note that the only equation which this changes in our new system is:

$$-\frac{\partial \hat{H}}{\partial K(t)} = \dot{\mu_1}(t) - \delta \cdot \mu_1(t) \tag{139}$$

which is now

$$-r_1 \cdot \mu_1(t) + \pi(t) \cdot \frac{\partial K(t)}{\partial K(t)} = \dot{\mu_1}(t) - \delta \cdot \mu_1(t) \tag{140}$$

or rather

$$-r_1 \cdot \mu_1(t) + \pi(t) = \dot{\mu_1}(t) - \delta \cdot \mu_1(t) \tag{141}$$

whereas it before was

$$-r_1 \cdot \mu_1(t) = \dot{\mu_1}(t) - \delta \cdot \mu_1(t) \tag{142}$$

Now, in the reference works I've consulted, this case is either mostly ignored (Daniel Liberzon's Calculus of Variations and Optimal Control Theory: A Concise Introduction, or Weber's Optimal Control Theory with Applications in Economics), or give fairly simplified examples (Optimal Control Theory Applications to Management Science and Economics; various lecture nots), saying that "In general, the solution of this problem is difficult".

I'm also not sure how to derive the shape of $\pi(t)$, though I think that it's given by $\dot{\mu} = 0$ when $K = 0$. Here, I'd appreciate a worked example, or the chance to ask a couple of questions to someone who works on this, though I already have some people in mind.

Interestingly, because the penalty term only kicks in once capital is zero, we know that the spending schedule for movement building, which we find out in (98), is also optimal in the case where the social movement starts with very few funds. In particular, it's optimal until the movement reaches $K = 0$, at which point I don't know what happens. We also know the spending schedule for direct giving up to a constant, but we don't know the constant (because, unlike in the non-transversality violation case, we can't do simulations to find that constant "empirically".)

Or, can we? I looked briefly into this, and there are some numerical methods to solve state-constrained problems, but I'm not optimistic.

B Numerical simulation details

B.1 Overview

We have determined the value of k_1 at all times (up to a constant c_1), as well the value of k_2. Now suppose we knew K and L at some point, for example at the present time t_0 i.e., $K(t_0), L(t_0)$. Then, we could also figure out $l_i(t_0)$, per (16) and (17):

$$k_1(t) = \left(\frac{1-q}{q \cdot w_2}\right)^{\frac{1}{\rho-1}} \cdot exp\left\{-\frac{\gamma_1}{\rho-1} \cdot t\right\} \cdot (L(t) \cdot l_1(t)) \tag{143}$$

$$L(t) = \left(\frac{q \cdot w_2}{1-q}\right)^{\frac{1}{\rho-1}} \cdot exp\left\{\frac{\gamma_1}{\rho-1} \cdot t\right\} \cdot \frac{k_1(t)}{l_1(t)} \tag{144}$$

Using $\alpha_1(t_0), \alpha_2(t_0), \sigma_1(t_0), \sigma_2(t_0), x_1(t_0), x_2(t_0)$ we can approximate the derivative, or instantaneous change of the state variables, $\dot{x}_1(t_0), \dot{x}_2(t_0)$ per their law of motion (3), and then approximate $x_i(t_0 \pm \epsilon) = x_i(t_0) \pm \epsilon \cdot \dot{x}_i(t_0)$. Our general approach to generate numerical approximations will be to use this approximation.

The method in which we start with the values at some initial point in time and then extrapolate them into the future is known as forward shooting. In contrast, the method in which we try to guess some final points in the future which, when extrapolated into the past hit our initial conditions is known as reverse shooting. Reverse shooting is known for being more stable, but in this instance it fails, perhaps because of floating point errors.

The code, in R, is inspired by previous Matlab code originally by Charles Jones, modified by Leopold Aschenbrenner and cleaned up by myself. Aschenbrenner's code can be found in this online repository: GitHub.com/ NunoSempere/ ReverseShooting, and my own code can be found in GitHub.com/ NunoSempere/ LaborCapitalAndTheOptimalGrowthOfSocialMovements .

This code makes use of the variable values from our example scenario in §4.1

$$
\begin{cases}
\eta = \text{Elasticy of spending} = 1.1 \\
\delta = \text{Hazard rate} = 0.005 = 0.5\% \\
\rho = \text{Substitution parameter from the CES production function} = -0.5 \\
q = \text{Share parameter from the CES production function} = 0.5 \\
r_1 = \text{Returns above inflation} = 0.06 = 6\% \\
r_2 = \text{Movement value drift and death rate} = -0.05 \\
\gamma_1 = \text{Change in participant contributions} = 0.03 = 3\% \\
\gamma_2 = \text{Change in the difficulty of recruiting} = 0.01 = 1\% \\
\lambda_2 = \text{Cobb-Douglas elasticity of movement building} = 0.5 \\
w_2 = \text{Initial participant contribution per unit of time} = 5000 \\
\beta_2 = \text{Constant inversely proportional to difficulty of recruiting} = 1,000 \\
\delta_2 = \text{Elasticity of movement growth} = 0.44
\end{cases}
\tag{145}
$$

To which we add β_2, w_2, because we care about absolute values, not just growth rates.

$$
\begin{aligned}
w_2 &= 5000 \\
\beta_2 &= 1
\end{aligned}
\tag{146}
$$

These factors correspond to each movement participant donating \$5000 per year, or 10% of a \$50.000 salary, and a team of five people being able to recruit 10 other people a year on a 20k budget (and mantaining those they have recruited previously.) Further work could be done in order to determine more accurate and realistic estimates. We also consider initial conditions:

$$K(t_0) = \texttt{x\_1\_init} = 10^{13} \tag{147}$$

$$L(t_0) = \texttt{x\_2\_init} = 10^4 \tag{148}$$

We also consider a parameters corresponding to our unknown constant c_1: `c1_forward_shooting`. This constant determines spending on direct work, and its value is such that decreasing it results in too little spending, and the movement accumulates money which is never spent. Conversely, increasing it results in the movement going bankrupt and acquiring infinite debt. However, its value is inexact, and will be a source of error. In particular, if we run simulations until time t, we don't know that the movement will not go bankrupt at some subsequent time, and hence `c1_forward_shooting` requires some guesswork. More specifically, if we select the maximum `c1_forward_shooting` such that K is positive at time t, we tend to find that $K \to -\infty$ shortly afterwards.

```
k1_forward_shooting = 10^(-8)
```

Finally, we decide on a step-size and on a time interval. We consider time intervals of 100, 500, 1000, 5000 and 10000 years.

```
stepsize = 0.1
first = 0
last = 10000
times_forward_shooting = seq(from=first, to=last, by=stepsize)
```

B.2 Problematic details

B.2.1 Floating point errors

Using a very small step size runs into floating point errors. Consider a stylized example:

```
options(digits=22)
dx <- 10^43
numsteps <- 10^7
stepsize <- 10^(-3)

## Example 1
x <- pi*1e+60
print(x)
for(i in c(1:numsteps)){
x <- x+dx*stepsize
}
x == pi*1e+60
# [1] TRUE
```

```
## Example 2
x <- pi*1e+60 + numsteps*stepsize*dx
x == pi*1e+60
# [1] FALSE
```

The two examples should give the same result, namely $pi * 1e + 60 + numsteps * stepsize * dx$, but don't. This is because in the first case, each step is so small that it is rounded off by the computer.

B.2.2 Reverse shooting

Perhaps because of floating point errors similar to the above, reverse shooting fails. Consider an stylized example

```
## Stylized forward shooting

x <- 0
for(i in c(1:30)){
  x <- x + 7^i
}

## Stylized reverse shooting

y <- x
for(i in c(30:1)){
  y <- y - 7^i
}
print(y)
# [1] -1227701488
```

Here, y should at the end be 0, but floating point errors ensure that it isn't. Given that our variables grow exponentially, we work with very large numbers and reverse shooting encounters similar errors and fails. Hence, we are restricted to using forward shooting.

C More Numerical Simulations

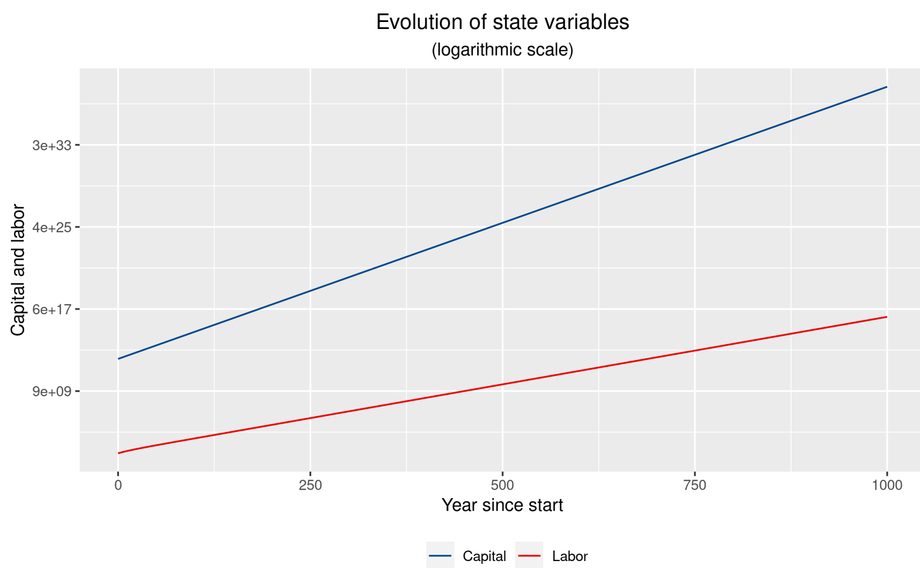
[This needs a better title] In this section, we will consider the variable values in the example in §4.1, in addition to:

$$\begin{aligned} r_2 &= -0.05 \\ w_2 &= 5000 \\ \beta_2 &= 1 \\ K(t_0) &= 10^{13} \\ L(t_0) &= 10^4 \end{aligned} \tag{149}$$

The r_2 value corresponds to a value drift (or death without replacement) rate of 5% per year. The w_2 value corresponds to a movement participant donating $5000 per year, or 10% of a $50.000 salary. The β_2 value corresponds to a team of five people being able to recruit 10 other people a year on a $20k budget (and maintaining those they have recruited previously with only an r_2 attrition rate.) The initial value for K corresponds to an absurdly large endowment and the initial value of L corresponds to 10k individuals broadly aligned with EA values. [For the moment, these values are chosen to make my life easier and not have to worry about transversality conditions]. Implementation details can be found in appendix §B.

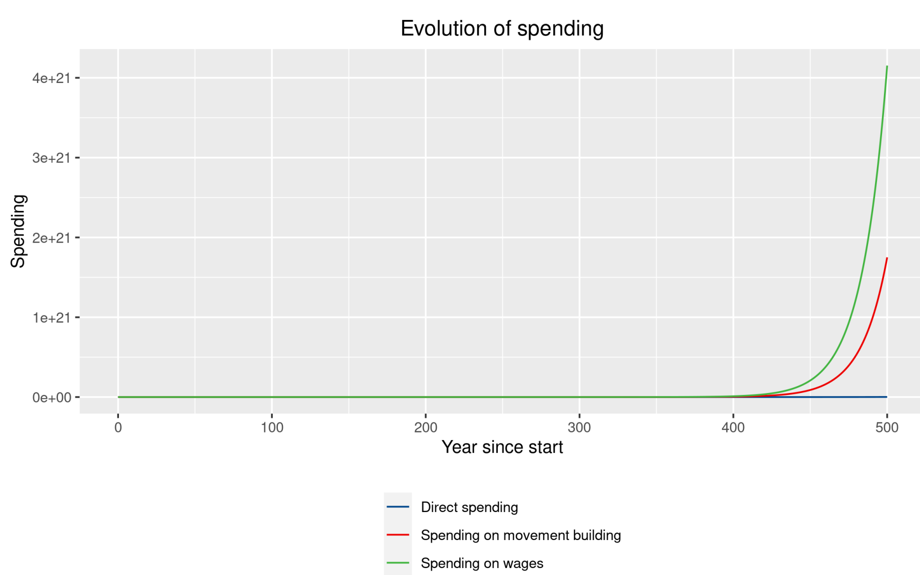
C.1 State variables

State variables, that is, capital and labor, grow at an exponential rate. Note that the scale is logarithmic, and that the notation 3e+33 denotes $3 \cdot 10^{33}$

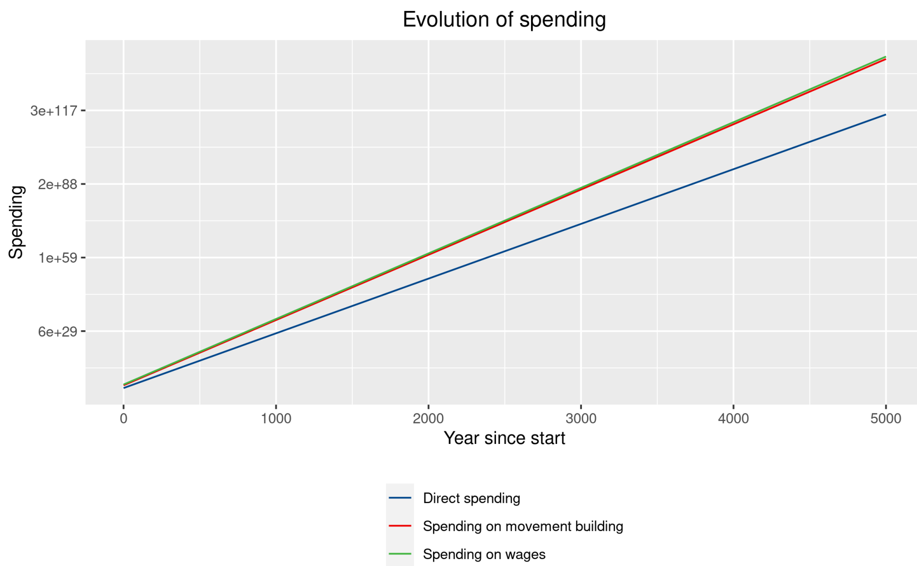


C.2 Spending rates

Spending also grows exponentially, with spending on movement building and on wages doing so at similar rates, and direct spending growing much more slowly.



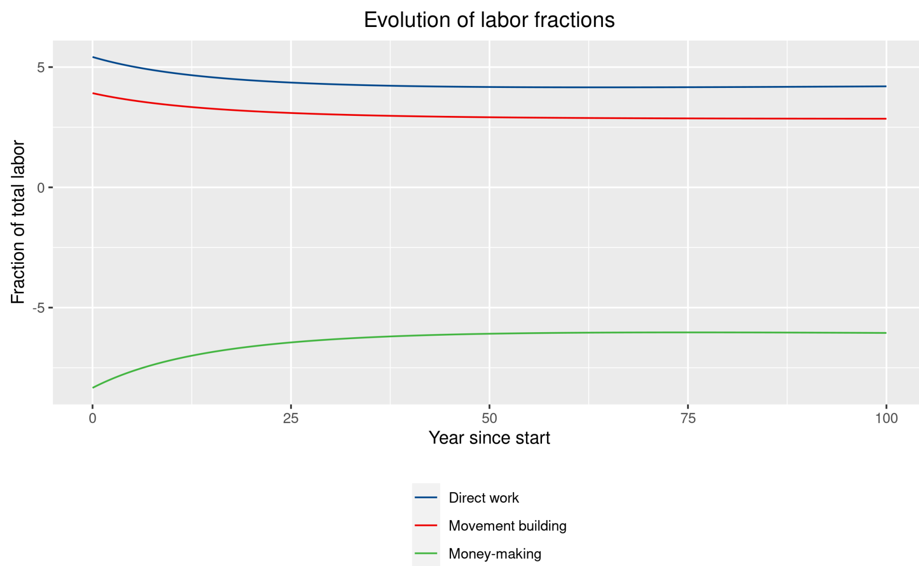
[Note to self: Add logarithmic to the subtitle]

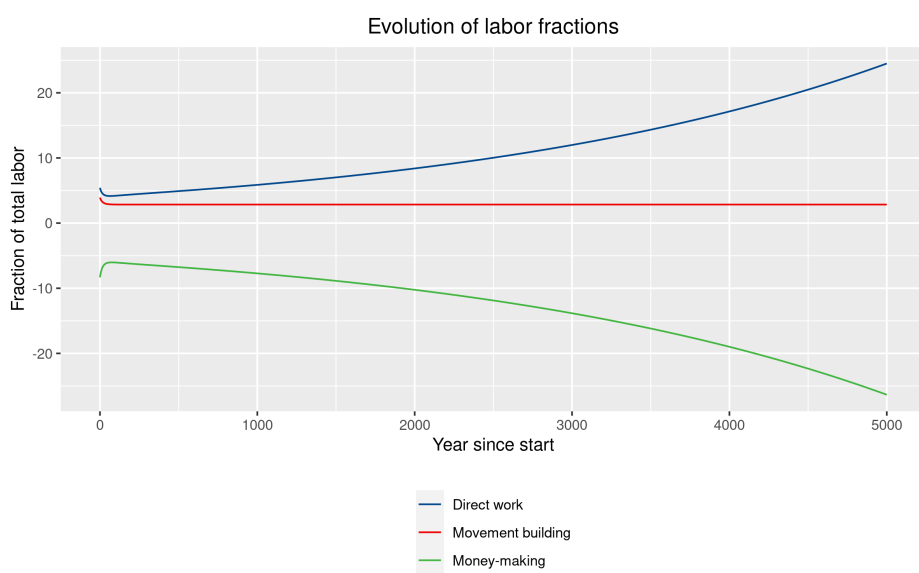


C.3 Allocation of labor

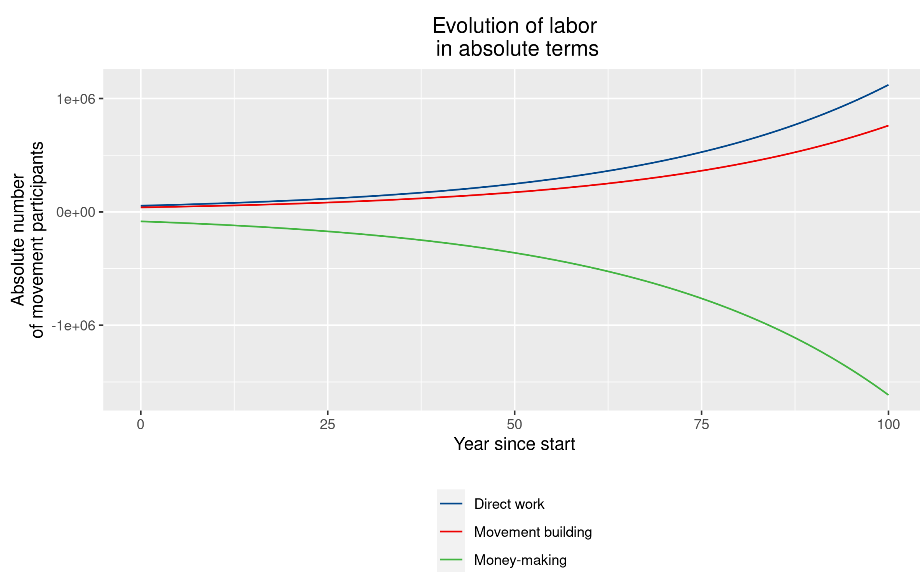
Overall, after initial behavior in the short term, all three forms of labor converge to their ultimate exponential growth forms. As a reminder, negative money-making denotes paying to hire.

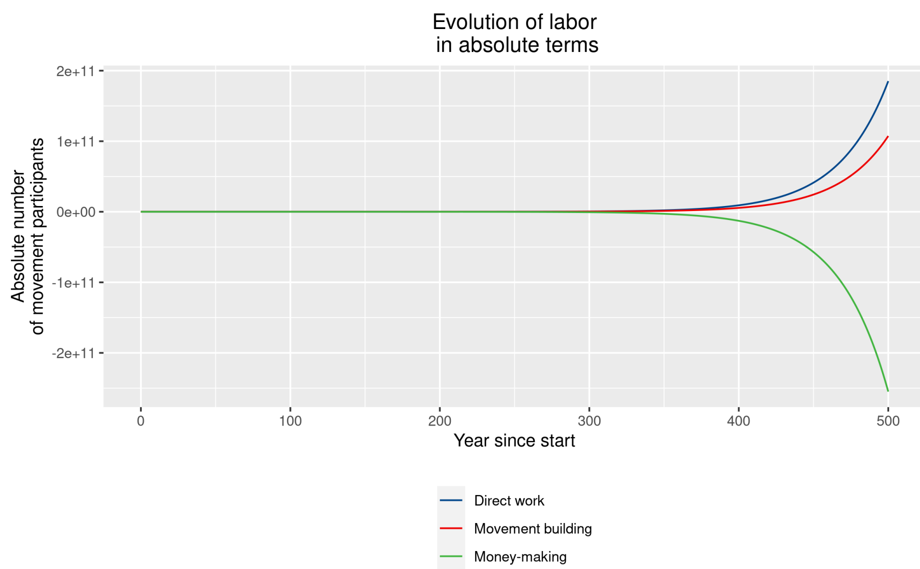
C.3.1 In relative terms

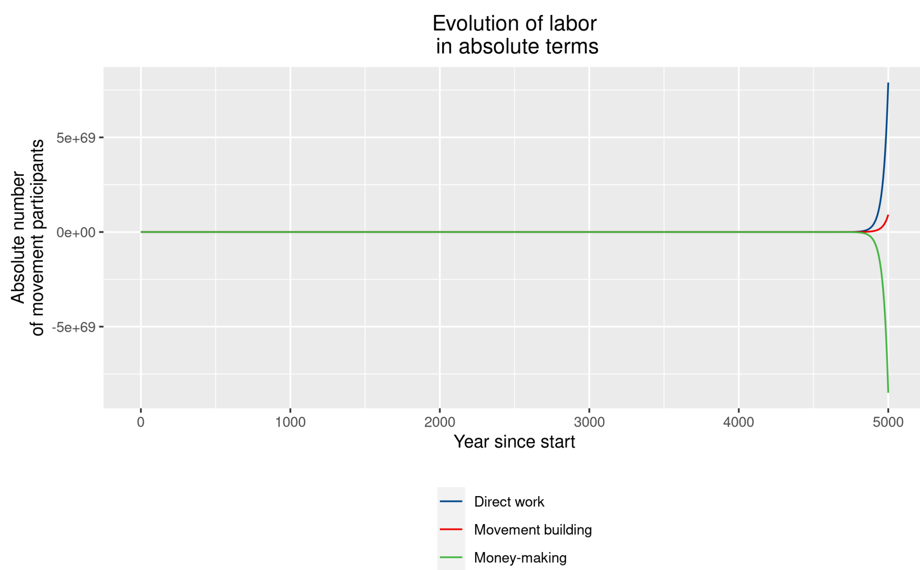




C.3.2 In absolute numbers







C.4 Overall behavior

The overall behavior can be described in terms of a short term non-exponential beginning, followed by a transition into the long-term stable growth paths. Those growth paths are characterized by money being spent on labor (either through hiring or through movement building), with this labor mostly being engaged in direct work.

Of course, because of the somewhat diminishing returns to either factor in our production functions, money is also being spent on direct work and labor also does some movement building. But this is a minority of spending and a minority of the work of labor. This is as we might expect from our asymptotic single-mindedness condition.

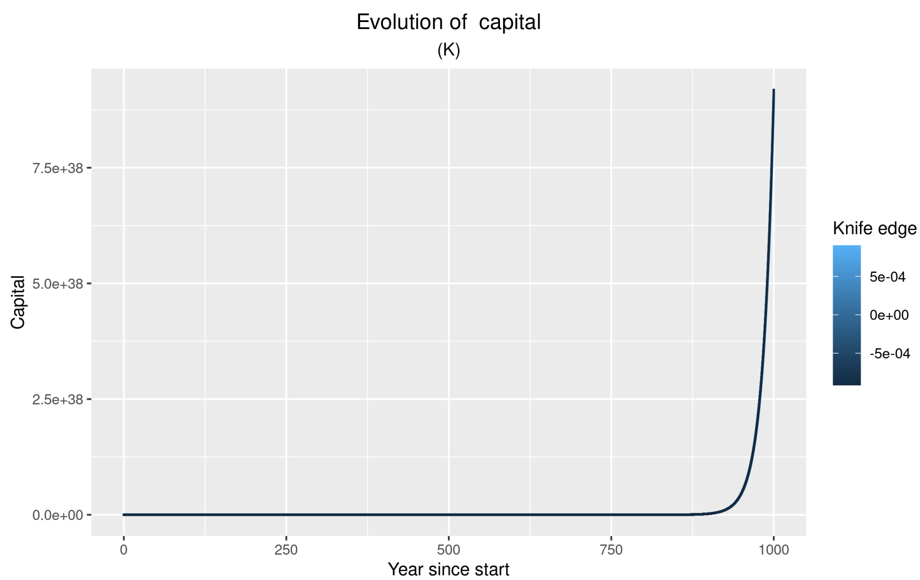
C.5 Behavior around the asymptotic single-mindedness knife-edge

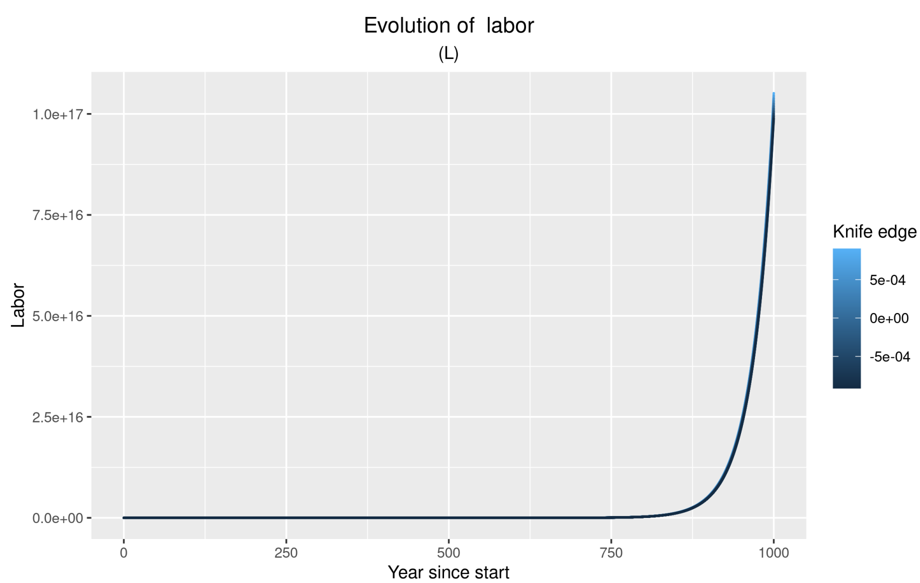
In Th.3, we proved that the ratio of movement building to direct work converged either to 0 or to ∞ with the passage of time. This depended of the behavior of the following constant:

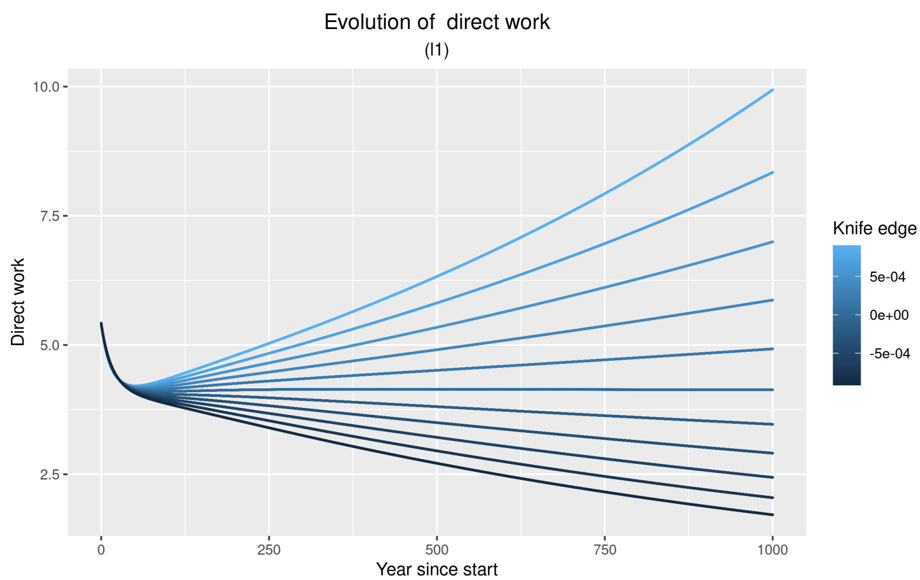
$$\begin{aligned} \text{Knife edge constant} = & \frac{\gamma_1}{\rho - 1} + \frac{r_1 - \delta}{\eta} \\ & + \max\left\{0, \frac{\gamma_1 \cdot (1 - \eta - \rho)}{\eta \cdot (\rho - 1)}\right\} \\ & - \max\left\{r_2, \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2}\right\} \end{aligned} \tag{150}$$

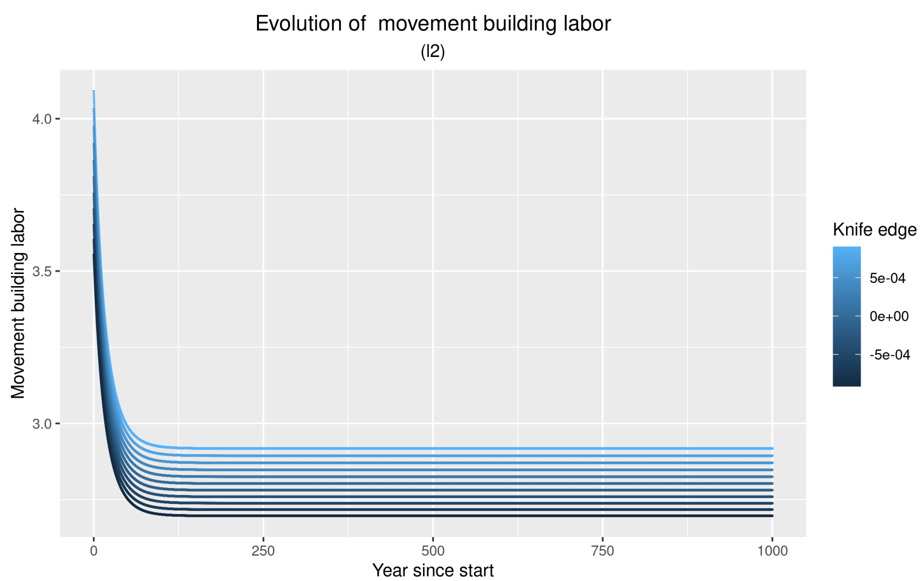
When doing simulations, we found that it was particularly convenient to increase or decrease this constant by manipulating δ, as the dependence between the knife edge constant and δ is straightforward and uncomplicated. Thus, in the following graphs, all variables remain the same except δ, which ranges from 0.0044, to 0.0064, in steps of 0.0002. When $\delta \approx 0.005392857143$, the knife edge constant is 0, and we land in the knife edge case.

Although capital and labor, K and L do change slightly as the knife edge constant changes, this is not apparent in the graph 1000 years out. The difference is more striking for the fractions of labor assigned to each capacity, and begins to be noticeable for spending at the end of the 1000 year period uncer consideration. Wages are negative because the social movement chooses to hire people, rather than to allocate its own members to money making. Similarly, remember that $l_3 = 1 - l_1 - l_2$, which can also be negative when the social movement chooses to hire people.

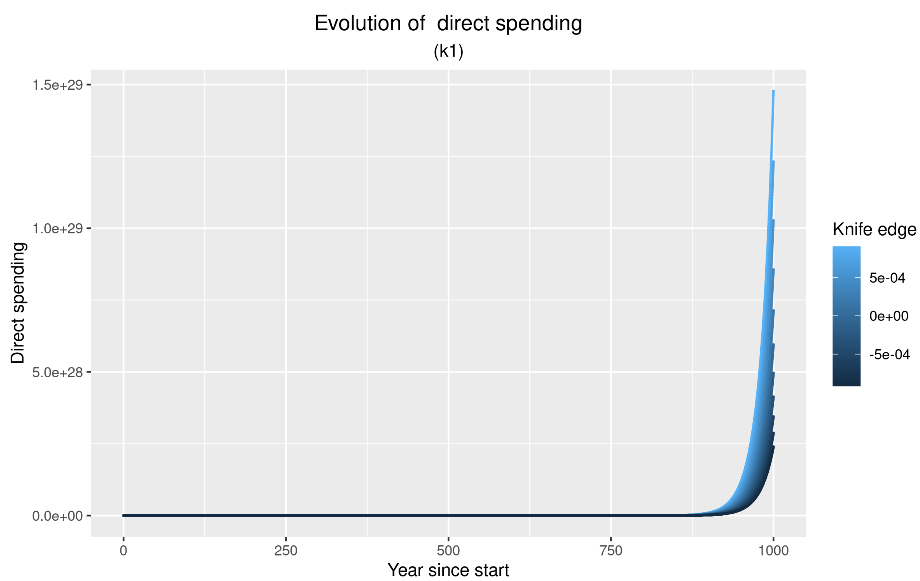

Evolution of capital
(K)
Capital
7.5e+38
5.0e+38
2.5e+38
0.0e+00
0
250
500
750
1000
Year since start
Knife edge
5e-04
0e+00
-5e-04


Evolution of labor
(L)
Labor
1.0e+17
7.5e+16
5.0e+16
2.5e+16
0.0e+00
0
250
500
750
1000
Year since start
Knife edge
5e-04
0e+00
-5e-04

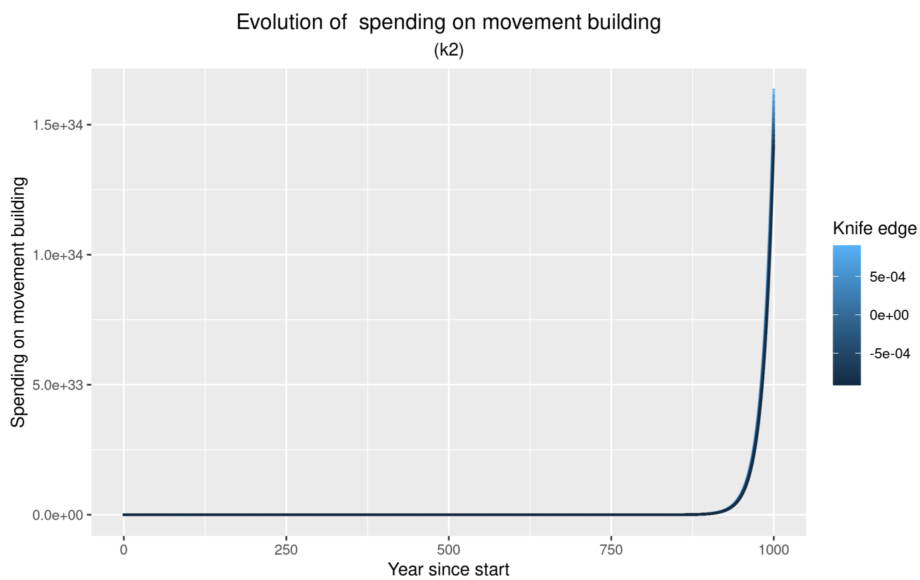

Evolution of direct work
(l1)
Direct work
10.0
7.5
5.0
2.5
0
250
500
750
1000
Year since start
Knife edge
5e-04
0e+00
-5e-04

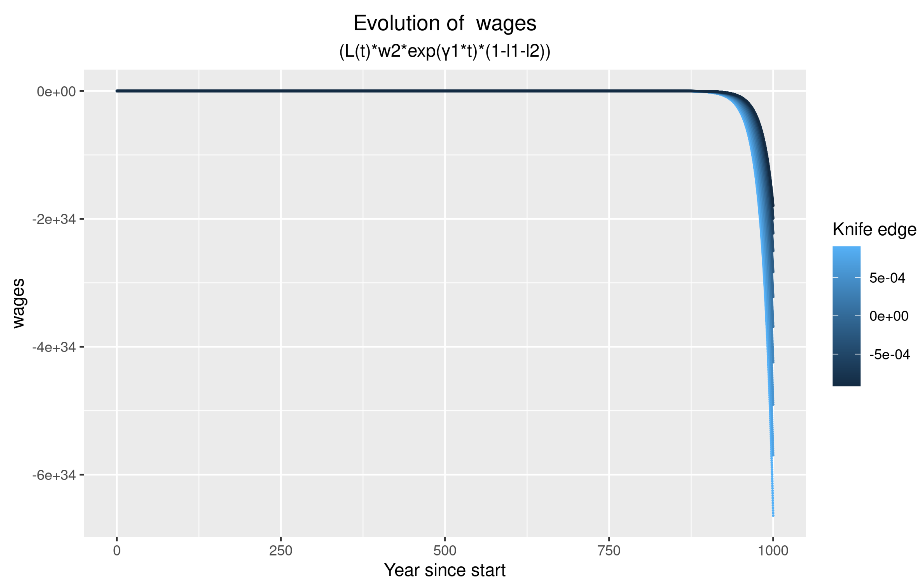

Evolution of movement building labor
(l2)
Movement building labor
4.0
3.5
3.0
0
250
500
750
1000
Year since start
Knife edge
5e-04
0e+00
-5e-04

Evolution of money-making/hired labor
(l3)
Money-making/hired labor
-5.0
-7.5
-10.0
0
250
500
750
1000
Year since start
Knife edge
5e-04
0e+00
-5e-04



Evolution of direct spending
(k1)
Direct spending
1.5e+29
1.0e+29
5.0e+28
0.0e+00
0
250
500
750
1000
Year since start
Knife edge
5e-04
0e+00
-5e-04





D Even More Numerical Simulations

[This needs a better title]

In this section, we will consider the following variable values:

$$
\begin{cases}
\eta = \text{Elasticy of spending} = 1.1 \\
\delta = \text{Hazard rate} = 0.008\dot{6} = 0.8\dot{6}\% \\
\rho = \text{Substitution parameter from the CES production function} = -0.5 \\
q = \text{Share parameter from the CES production function} = 0.5 \\
r_1 = \text{Returns above inflation} = 0.06 = 6\% \\
r_2 = \text{Returns to movement size} = -0.05 \\
\gamma_1 = \text{Change in participant contributions} = 0 = 3\% \\
\gamma_2 = \text{Change in the cost of recruiting} = 0 = 5\% \\
\lambda_2 = \text{Cobb-Douglas elasticity of movement building} = 0.5 \\
w_2 = \text{Initial participant contribution per unit of time} = 1000 \\
\beta_2 = \text{Constant inversely proportional to difficulty of recruiting} = 2 \\
\delta_2 = \text{Elasticity of movement growth} = 0.4 \\
r_2 = -0.05 \\
\beta_2 = 0.25 \\
K(t_0) = 2 \cdot 10^{10} \\
L(t_0) = 10^4
\end{cases}
\tag{151}
$$

By setting $\gamma_1 = \gamma_2 = 0$, we are essentially assuming that the rate of wage growth is equal to the rate of growth in productivity, and that other factors are negligible.

The r_2 value corresponds to a return to movement building size of -3% per year. On the one hand, value drift and death bring this rate down, on the other hand, a positive birth rate, increased productivity or a positive spontaneous idea-spreading rate would increase r_2. Overall, we estimate that the former factors outweigh the latter.

The w_2 value corresponds to a movement participant donating \$1000 per year, or 2.5% of a \$40.000 salary. The β_2 and δ_2 values correspond to 100 new movement participants, \$1M and 100 movement builders convincing 100 new movement participants each year, and retaining the ones already convinced. The initial value for K corresponds an initial endowment of \$20 billion, and the initial value of L corresponds to 10k individuals broadly aligned with EA values

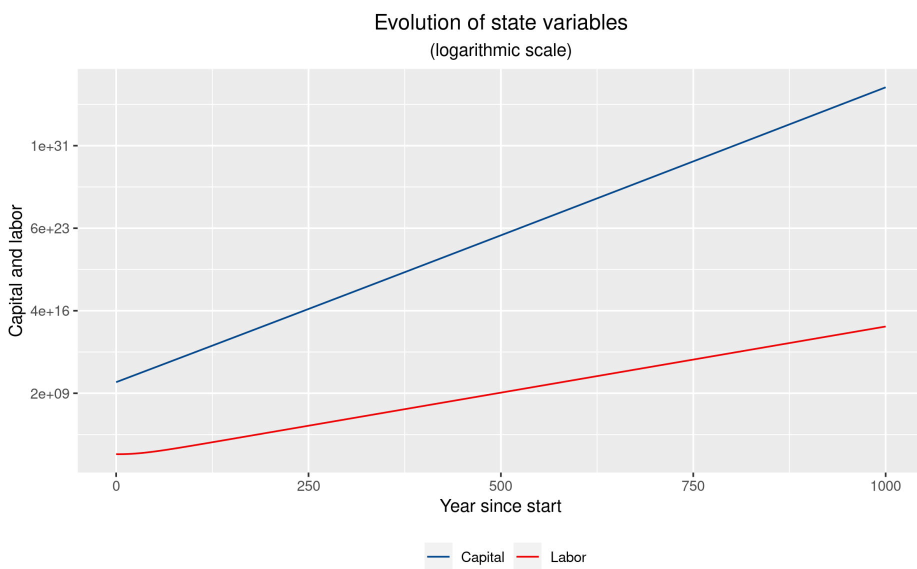
The implementation details of the simulation can be found in appendix

§B.

The hazard rate has been precisely chosen so that the movement lies on the knife edge. On the next section, we will explore the dynamics on the knife edge, and we will then see how they vary when the knife edge falls to either side.

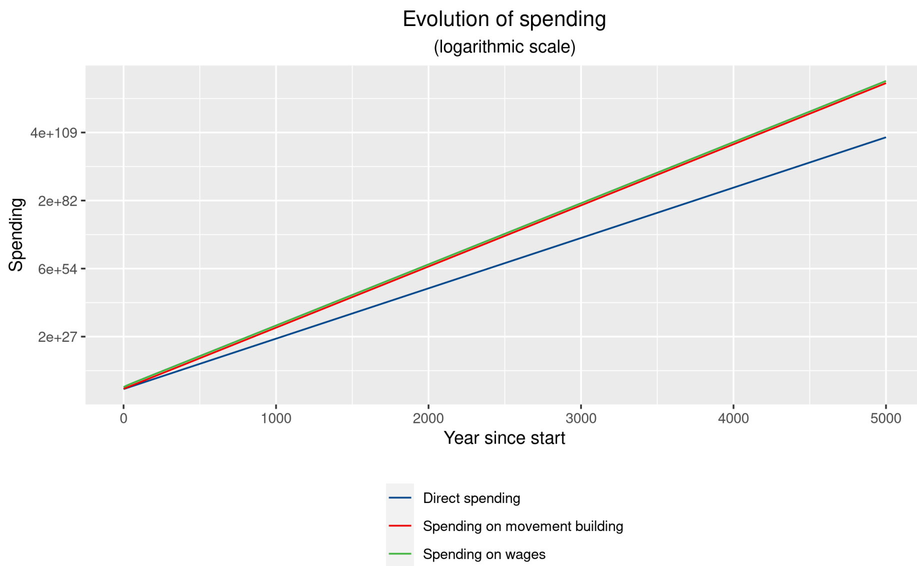
D.1 State variables

State variables, that is, capital and labor, grow at an exponential rate, after an initial period where movement size decreases. Note that the scale is logarithmic, and that the notation 3e+33 denotes $3 \cdot 10^{33}$



D.2 Spending rates

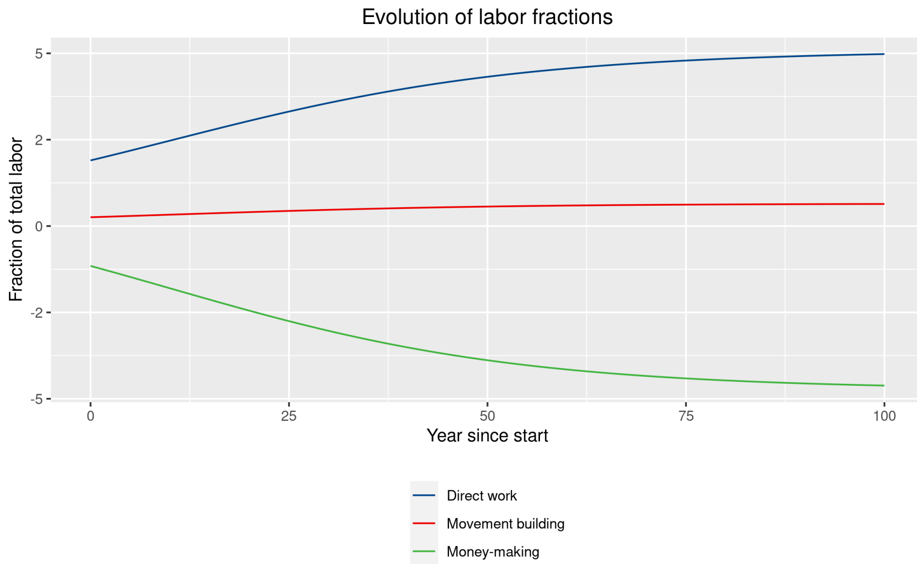
Spending also grows exponentially, with spending on movement building and on wages doing so at similar rates, and direct spending growing much more slowly.

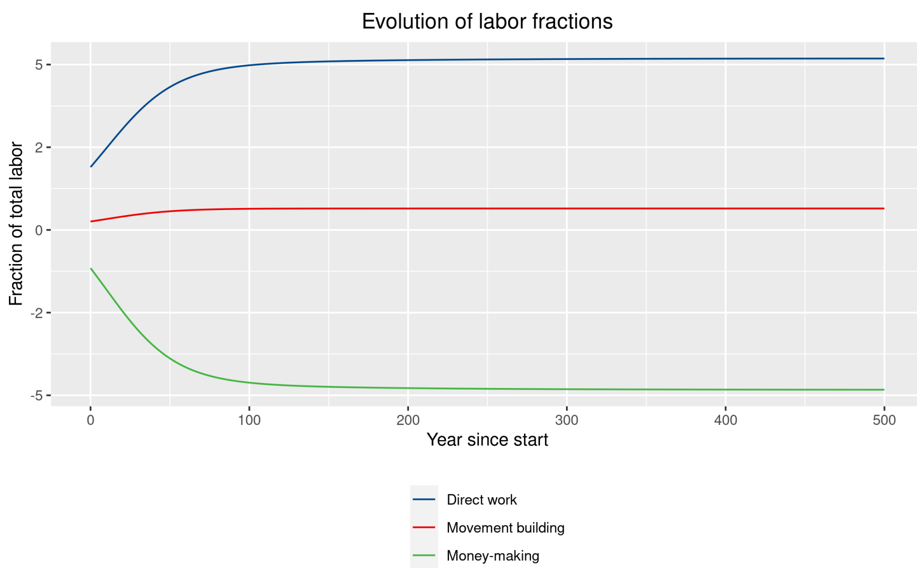


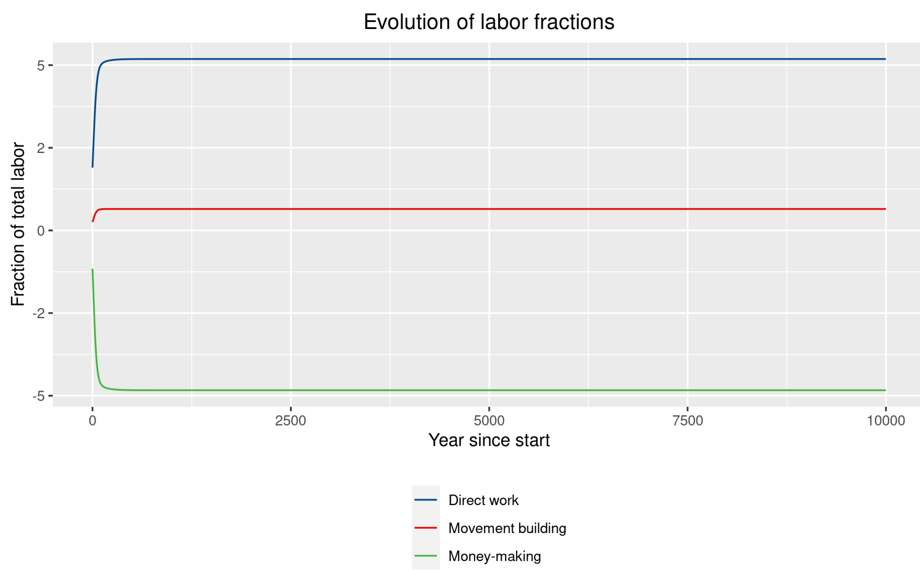
D.3 Allocation of labor

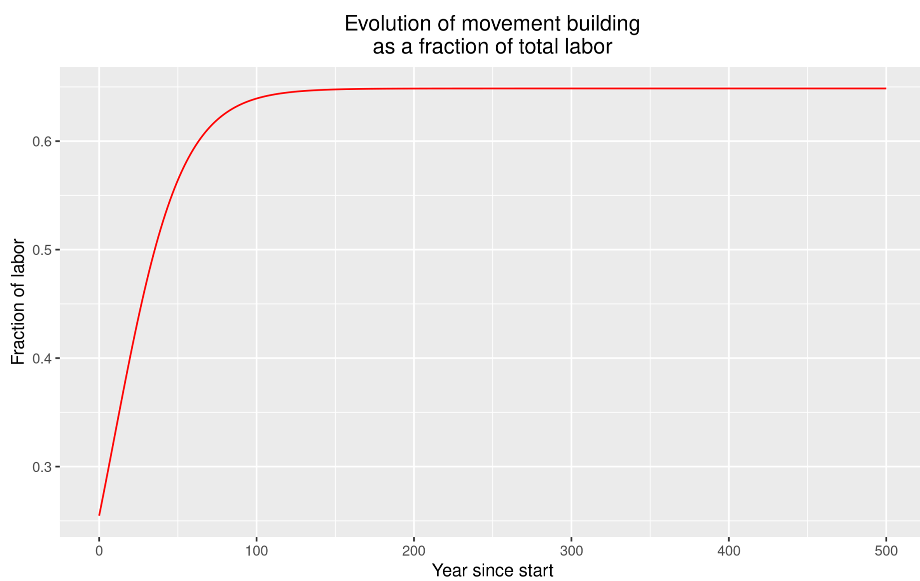
Overall, after initial behavior in the short term, all three forms of labor converge to their ultimate exponential growth forms. As a reminder, negative money-making denotes paying to hire.

D.3.1 In relative terms

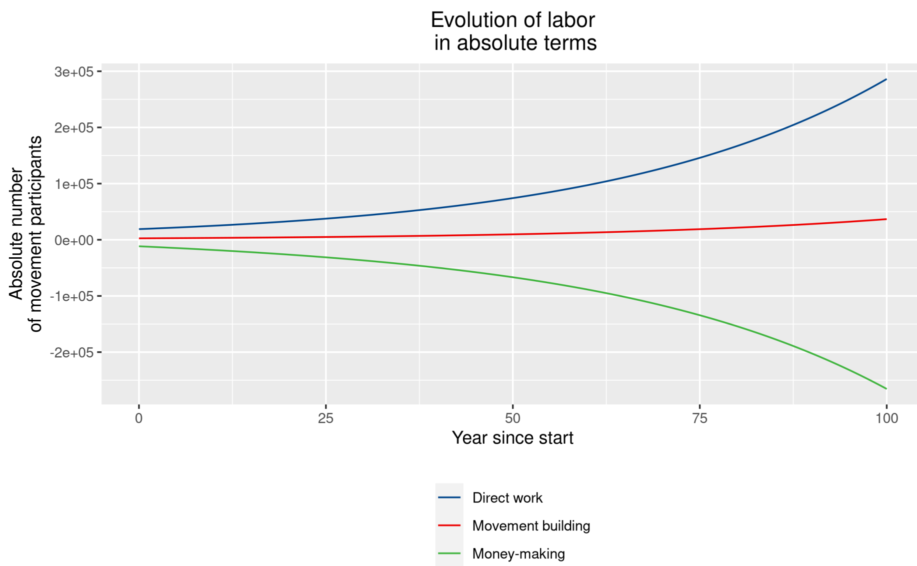


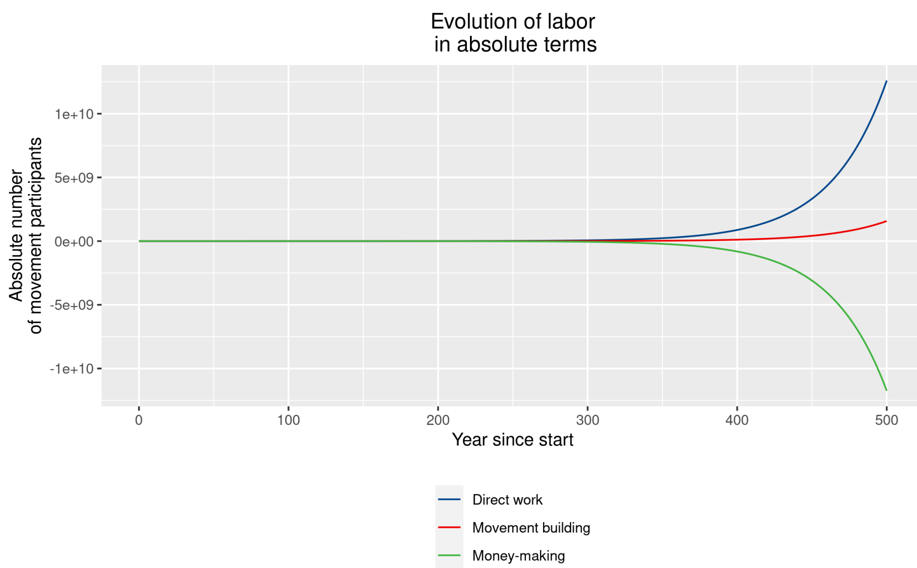


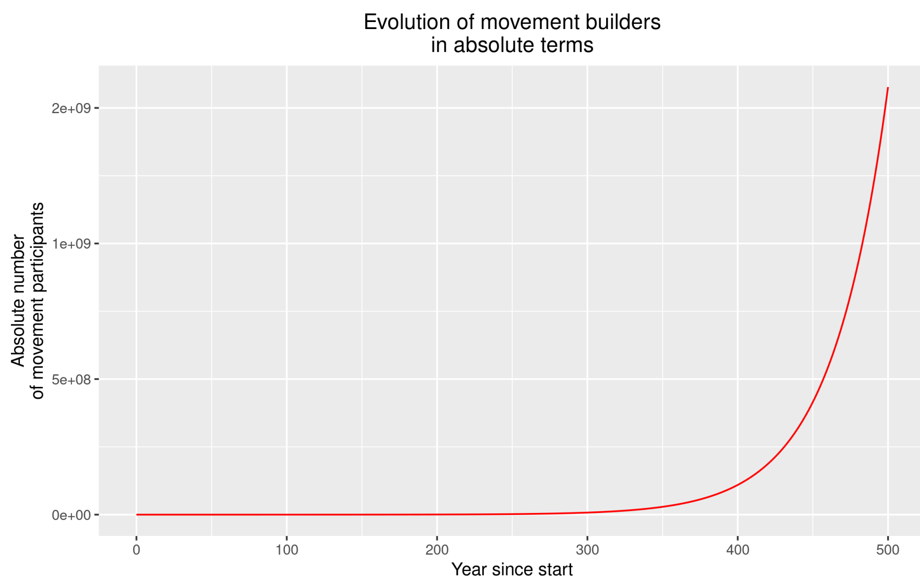

Evolution of labor fractions
Fraction of total labor
5
2
0
-2
-5
0
2500
5000
7500
10000
Year since start
Direct work
Movement building
Money-making

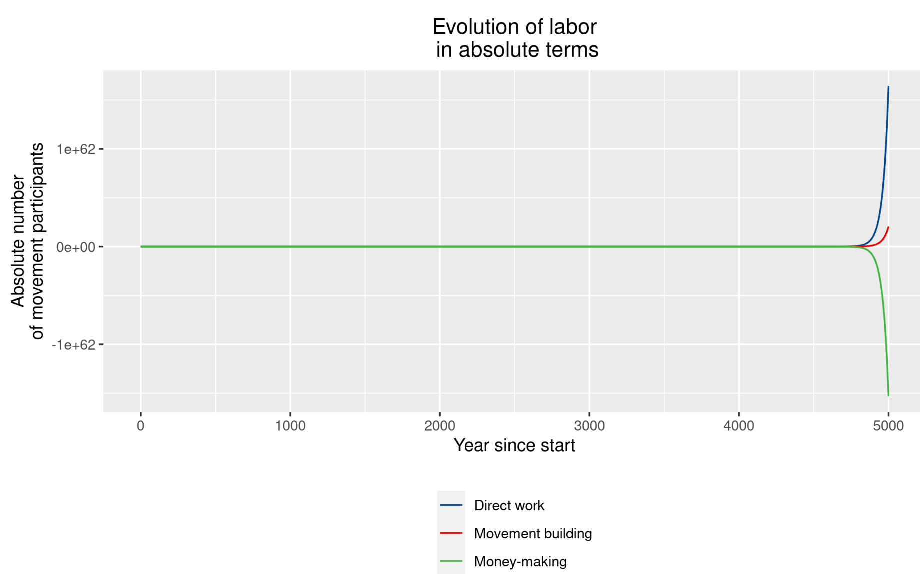

Evolution of movement building
as a fraction of total labor
Fraction of labor
0.6
0.5
0.4
0.3
0
100
200
300
400
500
Year since start

D.3.2 In absolute numbers









D.4 Overall behavior

The overall behavior can be described in terms of a short term non-exponential beginning, followed by a transition into the long-term stable growth paths. Those growth paths are characterized by money being spent on labor (either through hiring or through movement building), with this labor mostly being engaged in direct work. Interestingly, at the very beginning some labor is spent on money-making, which gradually transitions into paying for labor instead as time progresses.

Of course, because of the somewhat diminishing returns to either factor in our production functions, money is also being spent on direct work and labor also does some movement building. But this is a minority of spending and a minority of the work of labor.

D.5 Behavior around the asymptotic single-mindedness knife-edge

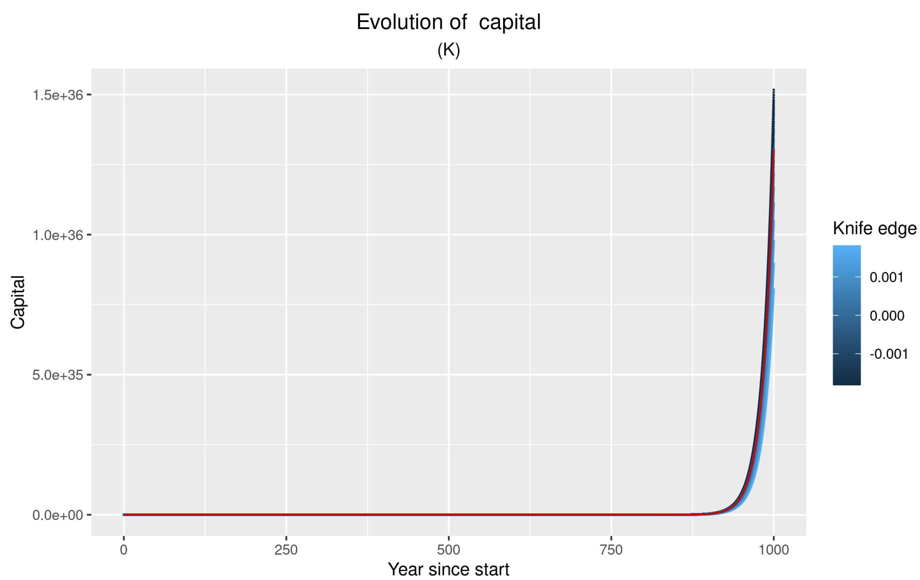
In Th.3, we proved that the ratio of movement building to direct work converged either to 0 or to ∞ with the passage of time. This depended of the behavior of the following constant:

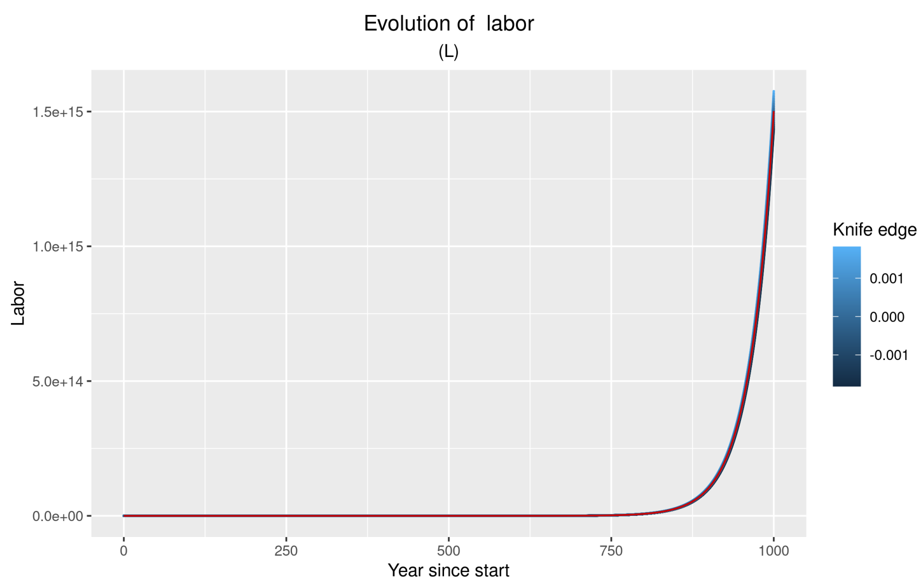
$$\begin{aligned}\text{Knife edge constant} = &\frac{\gamma_1}{\rho - 1} + \frac{r_1 - \delta}{\eta} \\ &+ \max\left\{0, \frac{\gamma_1 \cdot (1 - \eta - \rho)}{\eta \cdot (\rho - 1)}\right\} \\ &- \max\left\{r_2, \frac{\gamma_2 + \gamma_1 \cdot \delta_2 \cdot \lambda_2}{1 - \delta_2}\right\}\end{aligned} \tag{152}$$

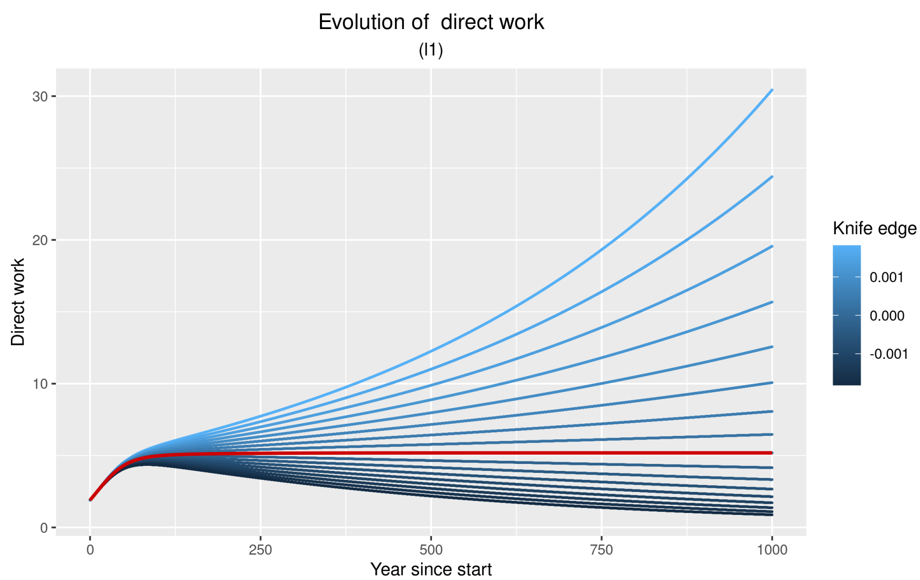
When doing simulations, we found that it was particularly convenient to increase or decrease this constant by manipulating δ, as the dependence between the knife edge constant and δ is straightforward and uncomplicated. Thus, in the following graphs, all variables remain the same except δ, which ranges from $0.00\dot{6}$, to $0.001\dot{6}$, in steps of 0.00025. When $\delta = 0.008\dot{6}$, the knife edge constant is 0, and we land in the knife edge case.

Although capital and labor, K and L do change slightly as the knife edge constant changes, this is not apparent in the graph 1000 years out. The difference is more striking for the fractions of labor assigned to each capacity, and begins to be noticeable for spending at the end of the 1000 year period under consideration. Wages are negative because the social movement chooses to hire people, rather than to allocate its own members to money making. Similarly, remember that $l_3 = 1 - l_1 - l_2$, which can also be negative when the social movement chooses to hire people.

The knife edge case, which we simulated in the previous section, is colored in red.

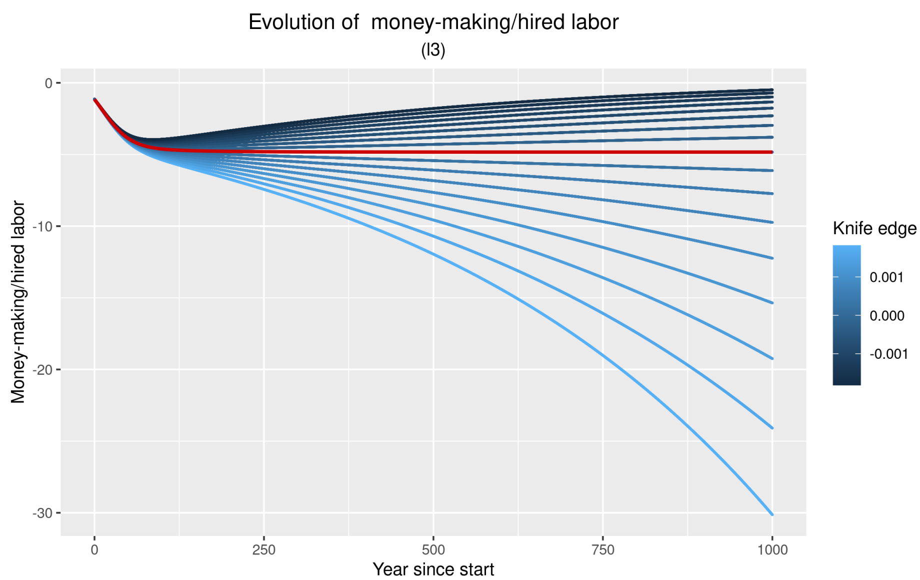

Evolution of capital
(K)
1.5e+36
1.0e+36
5.0e+35
0.0e+00
Capital
0
250
500
750
1000
Year since start
Knife edge
0.001
0.000
-0.001

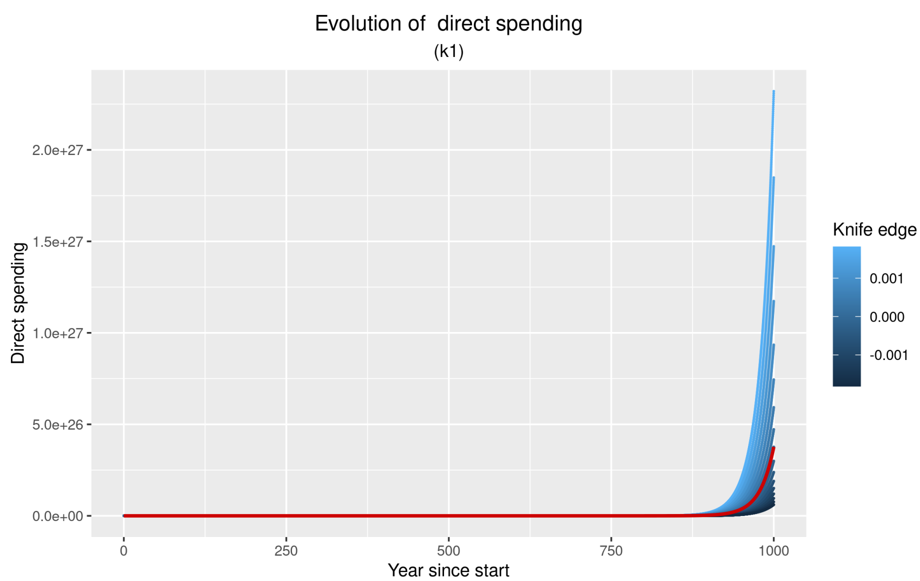

Evolution of labor
(L)
1.5e+15
1.0e+15
5.0e+14
0.0e+00
Labor
0
250
500
750
1000
Year since start
Knife edge
0.001
0.000
-0.001

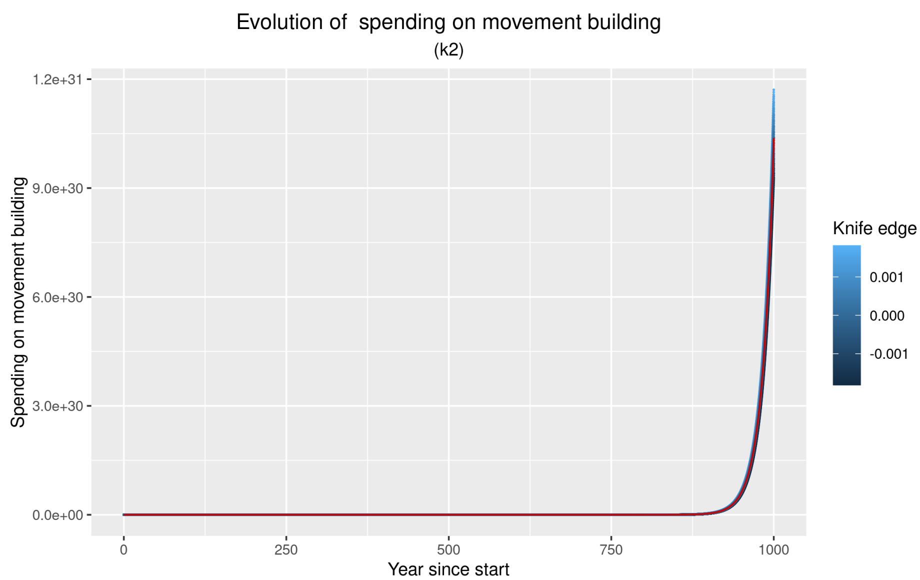


Evolution of direct work
(l1)
Direct work
30
20
10
0
0
250
500
750
1000
Year since start
Knife edge
0.001
0.000
-0.001

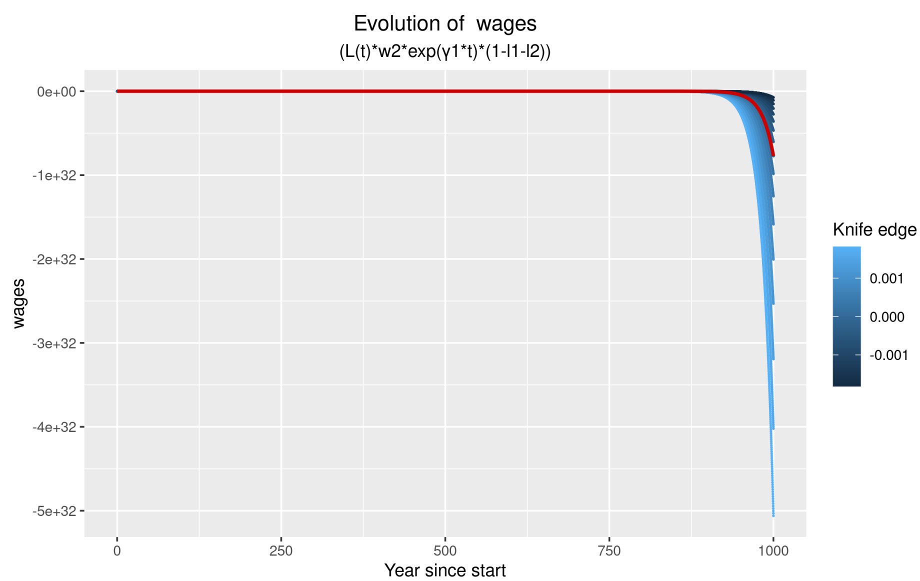
Evolution of movement building labor
(l2)
Movement building labor
0.7
0.6
0.5
0.4
0.3
0
250
500
750
1000
Year since start
Knife edge
0.001
0.000
-0.001


Evolution of money-making/hired labor
(l3)
Money-making/hired labor
Year since start
0
-10
-20
-30
0
250
500
750
1000
Knife edge
0.001
0.000
-0.001


Evolution of direct spending
(k1)
Direct spending
Year since start
2.0e+27
1.5e+27
1.0e+27
5.0e+26
0.0e+00
0
250
500
750
1000
Knife edge
0.001
0.000
-0.001



Evolution of spending on movement building
(k2)
Spending on movement building
1.2e+31
9.0e+30
6.0e+30
3.0e+30
0.0e+00
0
250
500
750
1000
Year since start
Knife edge
0.001
0.000
-0.001



Evolution of wages
(L(t)*w2*exp(γ1*t)*(1-l1-l2))
wages
0e+00
-1e+32
-2e+32
-3e+32
-4e+32
-5e+32
0
250
500
750
1000
Year since start
Knife edge
0.001
0.000
-0.001