Contents

Labor, Capital, and the Optimal Growth of Social Movements

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WORK IN PROGRESS

1 Introduction

Social movements such as "Effective Altruism" face the problem of optimal allocation of resources across time in order to maximize their desired impact. Much like states and other entities considered in the literature since (Ramsey, 1928) [1], they have the option to invest in order to give more later. However, unlike states, where population dynamics are usually considered exogenous, such agents also have the option of recruiting like-minded associates through movement building. For example, Bill Gates can recruit other ultra-rich people through the Giving Pledge, aspiring effective altruists can likewise spread their ideas, etc.

This paper models the optimal allocation of capital for a social movement between direct spending, investment, and movement building, as well as the optimal allocation of labor between direct workers, money earners, and movement builders. This research direction follows in the footsteps of (Trammell, 2020) [2], which considers a different facet of a related problem: the dynamics for a philanthropic funder who aims to provide public goods while having a lower discount rate than less patient partners.

The outline of this paper is as follows: §2 considers a social movement which starts out with a certain amount of money and a certain number of movement participants. This movement must then decide where to allocate their capital and labor. We work out some useful properties of the optimal solution, its long-term balanced growth rates, and some exact results about the optimal path. §3 presents the results from a numerical simulation. §4 concludes and outlines implications.

2 Movement building model

2.1 Setup

The variables under consideration are:

1. x_1, total capital, and x_2, total movement size (labor). r_1, the return rate on capital, and r_2, which will typically be negative and represent a decay rate, due to death, value drift, etc.

2. α_1, spending on direct work on a given instant, and α_2, the money spent on movement building on a given instant.

3. $\sigma_1, \sigma_2, \sigma_2$: the fraction of labor which works respectively on direct work, movement building, and money-making. $\sigma_1 + \sigma_2 + \sigma_3 = 1$, so well substitute $\sigma_3 = 1 - \sigma_1 - \sigma_2$ throughout.

4. $w_2 \cdot exp\{\gamma_1 t\}$: wages rising with economic growth, and $\beta_2 \cdot exp\{\gamma_2 t\}$: the changing difficulty of recruiting movement participants. γ_2 might be hypothesized to be negative, given that economic growth provides better outside options for potential movement participants, but empirically seems to be positive as movements learn how to do movement building better. For simplicity, we will consider these rates —γ_1 and γ_2— to be exogenous.

5. δ_2: movement building returns to scale

We are maximizing:

$$V(\alpha(\vec{t})) = \max_{\alpha(\vec{t})} \int_0^\infty e^{-\rho t} \cdot U(x(\vec{t}), \alpha(\vec{t}))dt \tag{1}$$

For utility and laws of motion:

$$U(x, \alpha) = \frac{(\alpha_1^{\lambda_1}(\sigma_1 x_2)^{1-\lambda_1})^{1-\eta}}{1-\eta} \tag{2}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} r_1 x_1 - \alpha_1 - \alpha_2 + x_2 \cdot w_2 \cdot exp\{\gamma_1 t\} \cdot (1 - \sigma_1 - \sigma_2) \\ r_2 x_2 + \beta_2 \cdot exp\{\gamma_2 t\} \cdot (\alpha_2^{\lambda_2} \cdot (\sigma_2 x_2)^{1-\lambda_2})^{\delta_2} \end{bmatrix} \tag{3}$$

under the constraints that

$$x_2 \geq 0 \wedge \alpha_i \geq 0 \wedge \sigma_1 + \sigma_2 \leq 1 \tag{4}$$

We define the Hamiltonian to be:

$$H := U + \mu_1 \cdot \dot{x}_1 + \mu_2 \cdot \dot{x}_2 \tag{5}$$

$$\begin{aligned} H = & \frac{(\alpha_1^{\lambda_1}(\sigma_1 x_2)^{1-\lambda_1})^{1-\eta}}{1-\eta} \\ & + \mu_1 \cdot (r_1 x_1 - \alpha_1 - \alpha_2 + x_2 \cdot w_2 \cdot exp\{\gamma_1 t\} \cdot (1 - \sigma_1 - \sigma_2)) \\ & + \mu_2 \cdot (r_2 x_2 + \beta_2 \cdot exp\{\gamma_2 t\} \cdot (\alpha_2^{\lambda_2} \cdot (\sigma_2 x_2)^{1-\lambda_2})^{\delta_2}) \end{aligned} \tag{6}$$

The transversality condition which our solution must comply with is:

$$\lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot x_i \cdot \mu_i = 0 \tag{7}$$

For convenience, $F_2 := \beta_2 \cdot (\alpha_2^{\lambda_2} \cdot (\sigma_2 x_2)^{1-\lambda_2})^{\delta_2}$. Note that $F_2 = \dot{x}_2 - r_2 x_2$

2.2 Variable ratios heuristic

Theorem 1. *Let the model described in (2.1) hold. Then, in the optimal path,*

$$\frac{(1-\lambda_1)}{\lambda_1} \cdot \frac{\alpha_1}{\sigma_1} = \frac{(1-\lambda_2)}{\lambda_2} \cdot \frac{\alpha_2}{\sigma_2} \tag{8}$$

We will provide two proofs, one using the derivation from an analysis of the Hamiltonian equations in §A, and another which considers the marginal values of these variables.

Proof (from analysis of the Hamiltonian equations in §A). By dividing (70) by (72) and (71) by (73), we conclude that:

$$\frac{\lambda_1}{\alpha_1} = \frac{1-\lambda_1}{\sigma_1 \cdot x_2 \cdot w_2 \cdot exp\{\gamma_1 t\}} \tag{9}$$

$$\frac{\lambda_2}{\alpha_2} = \frac{1-\lambda_2}{\sigma_2 \cdot x_2 \cdot w_2 \cdot exp\{\gamma_1 t\}} \tag{10}$$

and hence

$$\frac{(1-\lambda_1)}{\lambda_1} \cdot \frac{\alpha_1}{\sigma_1} = \frac{(1-\lambda_2)}{\lambda_2} \cdot \frac{\alpha_2}{\sigma_2} \tag{11}$$

□

We can also derive this result from the Euler equations, that is, just from the constraint that on the optimal path, the marginal value of moving labor and spending around should be equal to 0.

Proof (using the Euler equations).

$$\frac{\partial U}{\partial \$} = \frac{\partial U}{\partial labor} \cdot \frac{\partial labor}{\partial \$}_{\text{bought out of money-making}} \tag{12}$$

$$\frac{\partial people}{\partial \$}_{\text{through movement building}} = \frac{\partial labor}{\partial labor} \cdot \frac{\partial labor}{\partial \$}_{\text{bought out of money-making}} \tag{13}$$

Equation (12) reads as "the *marginal* money-maker should produce as much value by making money and directly donating their earnings as by working directly." Equation (12) reads as "the *marginal* money-maker should create as many movement participants by making money and donating their earnings to movement building as by working on movement building themselves." Otherwise, we could move direct workers or movement builders towards money-making, or vice-versa.

From (3) and (5), the model definition, these two equations develop into:

$$\lambda_1 \cdot (1-\eta) \cdot \frac{U}{\alpha_1} = \left((1-\lambda_1) \cdot (1-\eta) \cdot \frac{U}{\sigma_1 \cdot x_2}\right) \cdot \left(\frac{1}{w_2 \cdot exp\{\gamma_1 \cdot t\}}\right) \tag{14}$$

$$\lambda_2 \cdot \delta_2 \cdot \frac{F_2}{\alpha_2} = \left((1-\lambda_2) \cdot \delta_2 \cdot \frac{F_2}{\sigma_2 \cdot x_2}\right) \cdot \left(\frac{1}{w_2 \cdot exp\{\gamma_1 \cdot t\}}\right) \tag{15}$$

Which simplify into

$$\frac{\lambda_1}{\alpha_1} = \frac{1-\lambda_1}{\sigma_1 \cdot x_2 \cdot w_2 \cdot exp\{\gamma_1 t\}} \tag{16}$$

$$\frac{\lambda_2}{\alpha_2} = \frac{1-\lambda_2}{\sigma_2 \cdot x_2 \cdot w_2 \cdot exp\{\gamma_1 t\}} \tag{17}$$

i.e., (9) and (10), from which (8) follows by isolation of the $x_2 \cdot w_2 \cdot exp\{\gamma_1 t\}$ term:

$$\frac{(1-\lambda_1)}{\lambda_1} \cdot \frac{\alpha_1}{\sigma_1} = \frac{(1-\lambda_2)}{\lambda_2} \cdot \frac{\alpha_2}{\sigma_2} \tag{18}$$

□

We can understand this equation as a convenient necessary but not sufficient heuristic, such that a spending schedule which doesn't satisfy it cannot be optimal, because one would be able to obtain a better outcome by allocating marginal capital or labor differently. This heuristic can also be expressed in simpler terms by abstracting the λ_i away:

$$\frac{\alpha_1}{\sigma_1} \cdot = \text{constant} \cdot \frac{\alpha_2}{\sigma_2} \tag{19}$$

respectively

$$\frac{\alpha_1}{\sigma_1 \cdot x_2} \cdot = \text{constant} \cdot \frac{\alpha_2}{\sigma_2 \cdot x_2} \tag{20}$$

2.3 Results

2.3.1 Balanced growth rates

The balanced growth rates for our variables are derived in §§A.1 through A.4. They are :

$$g_{x_2} = \frac{\gamma_2 + \delta_2 \lambda_2 \gamma_1}{1-\delta_2} \tag{21}$$

$$g_{\alpha_2} = g_{x_2} + \gamma_1 = \left[\frac{\gamma_2 + \delta_2 \lambda_2 \gamma_1}{1-\delta_2}\right] + \gamma_1 \tag{22}$$

$$g_{\sigma_2} = 0 \tag{23}$$

$$g_{\alpha_1} = \frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta} \cdot \gamma_1 \tag{24}$$

$$g_{\sigma_1} = \frac{r - \rho}{\eta} - \left(\frac{(1-\eta)(1-\lambda_1)}{\eta} + 1\right) \cdot \gamma_1 - g_{x_2} \tag{25}$$

subject to the transversality conditions:

$$\begin{cases} g_{\alpha_1} < r_1 \\ g_{\alpha_2} < r_1 \\ r_2 + \gamma_1 < r_1 \end{cases} \tag{26}$$

2.3.2 Asymptotic Quasi-Ponzi

Theorem 2. *Let the model described in (2.1) hold. Then, in the optimal path, in almost all cases:*

$$\frac{\sigma_1}{\sigma_1 + \sigma_2} \to 0 \tag{27}$$

$$\frac{\alpha_1}{\alpha_1 + \alpha_2} \to 0 \tag{28}$$

Proof. Recall (8):

$$\frac{(1-\lambda_1)}{\lambda_1} \cdot \frac{\alpha_1}{\sigma_1} = \frac{(1-\lambda_2)}{\lambda_2} \cdot \frac{\alpha_2}{\sigma_2} \tag{29}$$

Per results on the previous section, (2.3.1), $g_{\sigma_2} = 0$. Further, we know that $g_{\sigma_1} \leq 0$; it can't be the case that $g_{\sigma_1} > 0$ because then σ_1, the fraction of movement building allocated to direct work would eventually exceed 100%.

In particular, $g_{\sigma_1} < 0$ unless we're on the knife edge case where

$$\frac{r-\rho}{\eta} - \left(\frac{(1-\eta)(1-\lambda_1)}{\eta} + 1 \right) \cdot \gamma_1 = \frac{\gamma_2 + \delta_2 \lambda_2 \gamma_1}{1 - \delta_2} \tag{30}$$

So, unless σ_1 is on that knife edge case, $\sigma_1 \to 0$ and because σ_2 converges to a nonzero constant in almost all cases:

$$\frac{\sigma_1}{\sigma_1 + \sigma_2} \to 0 \tag{31}$$

Per (8), $g_{\alpha_2} - \cancel{g_{\sigma_2}} = g_{\alpha_1} - g_{\sigma_1}$, and because $g_{\sigma_1} < 0$ in almost all cases, $g_{\alpha_2} > g_{\alpha_1}$. Because α_2 then grows faster than α_2, this directly implies:

$$\frac{\alpha_1}{\alpha_1 + \alpha_2} \to 0 \tag{32}$$

□

This is reminiscent of a Ponzi scheme or of a multi-level-marketing scheme, because in the limit, most participants don't do direct-work either. In section §3 we will notice that this behavior may hold in the limit, but doesn't hold in the near-term.

2.3.3 Exact spending schedules

Theorem 3. *Let the model described in (2.1) hold. Then, in the optimal path,*

$$\alpha_1^{\eta} = \frac{\lambda_1}{k_1 \cdot exp\{(\rho - r_1)t\}} \cdot \left(\frac{1-\lambda_1}{\lambda_1 \cdot w_2 \cdot exp\{\gamma_1 t\}} \right)^{(1-\lambda_1)(1-\eta)} \tag{33}$$

$$\alpha_2^{1-\delta_2} = \frac{w_2 \cdot exp\{\gamma_1 \cdot t\}}{r_1 - \gamma_1 - r_2} \cdot \delta_2 \cdot \lambda_2 \cdot \beta_2 \cdot exp\{\gamma_2 t\} \cdot \left(\frac{1-\lambda_2}{\lambda_2 \cdot w_2 \cdot exp\{\gamma_1 t\}} \cdot \right)^{\delta_2 \cdot (1-\lambda_2)} \tag{34}$$

where k_1 is minimized subject to the constraint that $\lim_{t \to \infty} x_1 \geq 0$

Proof. See §A.5 □

One conclusion is that the value drift rate, (as long as it's low enough and satisfies $r_2 + \gamma_1 < r_1$, per (26), doesn't change the growth rate of spending on movement building, but affects it as a one time multiplicative ratio.

2.4 Example values

2.4.1 Example 1. $\eta = 1.1, \gamma_1 = 0.03, \delta_2 = 0.44$

$$\begin{cases} \eta = \text{Elasticy of spending} = 1.1 \\ \rho = \text{Hazard rate} = 0.005 = 0.5\% \\ r_1 = \text{Returns above inflation} = 0.06 = 6\% \\ \gamma_1 = \text{Change in participant contributions} = 0.03 = 3\% \\ \gamma_2 = \text{Change in the difficulty of recruiting} = 0.01 = 1\% \\ w_2 = \text{Average participant contribution per unit of time} = 0.5 \\ \beta_2 = \text{Constant inversely proportional to difficulty of recruiting} = 1,000 \\ \lambda_1 = \text{Coub-Douglas elasticity of direct work and direct spending} = 0.5 \\ \lambda_2 = \text{Coub-Douglas elasticity of movement building} = 0.5 \\ \delta_2 = \text{Elasticity of movement growth} = 0.44 \end{cases} \tag{35}$$

$$\begin{aligned} g_{x_2} &= \frac{\gamma_2 + \delta_2\lambda_2\gamma_1}{1-\delta_2} \\ &= \frac{0.01 + 0.5 \cdot 0.44 \cdot 0.03}{1 - 0.44} \\ &= 0.0296 = 2.96\% \end{aligned} \tag{36}$$

$$\begin{aligned} g_{\alpha_2} &= g_{\sigma_2} + g_{x_2} + \gamma_1 = 0 + g_{x_2} + \gamma_1 \\ &= 0.0296 + 0.03 \\ &= 0.0596 = 5.96\% \end{aligned} \tag{37}$$

$$\begin{aligned} g_{\alpha_1} &= \frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta} \cdot \gamma_1 \\ &= \frac{0.06 - 0.005}{1.1} - \frac{(1-1.1)(1-0.5)}{1.1} \cdot 0.03 \\ &\approx 0.05136 = 5.136\% \end{aligned} \tag{38}$$

$$\begin{aligned} g_{\sigma_1} &= g_{\alpha_1} - g_{x_2} - \gamma_1 \\ &= 0.05136 - 0.0296 - 0.03 \\ &= -0.00824 = -0.824\% \end{aligned} \tag{39}$$

2.4.2 Example 2. $\eta = 0.9, \gamma_1 = 0.03, \delta_2 = 0.44$

$$\begin{cases} \eta = \text{Elasticy of spending} = 0.9 \\ \rho = \text{Hazard rate} = 0.005 = 0.5\% \\ r_1 = \text{Returns above inflation} = 0.06 = 6\% \\ \gamma_1 = \text{Change in participant contributions} = 0.03 = 3\% \\ \gamma_2 = \text{Change in the difficulty of recruiting} = 0.01 = 1\% \\ w_2 = \text{Average participant contribution per unit of time} = 0.5 \\ \beta_2 = \text{Constant inversely proportional to difficulty of recruiting} = 1,000 \\ \lambda_1 = \text{Coub-Douglas elasticity of direct work and direct spending} = 0.5 \\ \lambda_2 = \text{Coub-Douglas elasticity of movement building} = 0.5 \\ \delta_2 = \text{Elasticity of movement growth} = 0.44 \end{cases} \tag{40}$$

$$\begin{aligned} g_{x_2} &= \frac{\gamma_2 + \delta_2 \lambda_2 \gamma_1}{1 - \delta_2} \\ &= \frac{0.01 + 0.5 \cdot 0.44 \cdot 0.03}{1 - 0.44} \\ &= 0.0296 = 2.96\% \end{aligned} \tag{41}$$

$$\begin{aligned} g_{\alpha_2} &= g_{\sigma_2} + g_{x_2} + \gamma_1 = 0 + g_{x_2} + \gamma_1 \\ &= 0.0296 + 0.03 \\ &= 0.0596 = 5.96\% \end{aligned} \tag{42}$$

$$\begin{aligned} g_{\alpha_1} &= \frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta} \cdot \gamma_1 \\ &= \frac{0.06 - 0.005}{0.9} - \frac{(1-0.9)(1-0.5)}{0.9} \cdot 0.03 \\ &\approx 0.0594 = 5.94\% \end{aligned} \tag{43}$$

$$\begin{aligned} g_{\sigma_1} &= g_{\alpha_1} - g_{x_2} - \gamma_1 \\ &= 0.0594 - 0.0296 - 0.03 \\ &= -0.0002 = -0.02\% \end{aligned} \tag{44}$$

2.4.3 Comparison with a rule of thumb allocation

Take a rule of thumb allocation, where $\sigma_1 = \sigma_2 = 0.5$, and the movement spends 1% of its capital per year, which then grows at 5% per year (i.e., $g_{\alpha_1} = g_{\alpha_2} = g_{x_1} = 0.05$).

For Example 1. ($\eta = 1.1$) Let $\lambda_1 = \lambda_2 = 0.5$, and in general let all the variables be as in the $\eta = 1.1$ example. Then for our rule of thumb allocation, the growth rate for x_2 is:

$$g_{x_2} = \gamma_2 + \delta_2 \cdot (\lambda_2 \cdot g_{\alpha_2} + (1 - \lambda_2) \cdot (g_{\sigma_2} + g_{x_2})) \tag{45}$$

$$g_{x_2} = 0.01 + 0.5 \cdot (0.5 \cdot 0.05 + 0.44 \cdot (0 + g_{x_2})) \tag{46}$$

$$g_{x_2} = 0.0288462 \approx 0.0288 \tag{47}$$

Then consider the growth of U in our rule of thumb allocation:

$$U(x, \alpha) = \frac{(\alpha_1^{\lambda_1} (\sigma_1 x_2)^{1-\lambda_1})^{1-\eta}}{1 - \eta} \tag{48}$$

$$g_U = (1 - \eta) \cdot (\lambda_1 \cdot g_{\alpha_1} + (1 - \lambda_1) \cdot (g_{\sigma_1} + g_{x_2})) \tag{49}$$

$$g_U = (1 - 1.1) \cdot (0.5 \cdot 0.05 + (1 - 0.5) \cdot (0 + 0.0288)) = -0.00394 \tag{50}$$

In contrast the growth of U in our first example is equal to:

$$g_U = (1-1.1) \cdot (0.5 \cdot 0.0594 + (1-0.5) \cdot (-0.00824 + 0.0296)) \approx -0.004038 \tag{51}$$

Note that when $\eta > 1$, the utility term is always negative, and thus a faster decrease is preferable.

For Example 2. ($\eta = 0.9$) Using the same reasoning as before, for the rule of thumb:

$$g_{x_2} \approx 0.0288 \tag{52}$$

$$g_U = (1 - 0.9) \cdot (0.5 \cdot 0.05 + (1 - 0.5) \cdot (0 + 0.0288)) = 0.00394 \tag{53}$$

In comparison, in the optimal path, the growth rate for U is:

$$g_U = (1 - 0.9) \cdot (0.5 \cdot 0.0594 + (1 - 0.5) \cdot (-0.0002 + 0.0296)) = 0.00444 \tag{54}$$

Note that when $\eta < 1$, the utility term is positive, and so higher growth in utility is preferable.

3 Numerical simulations

3.1 Setup

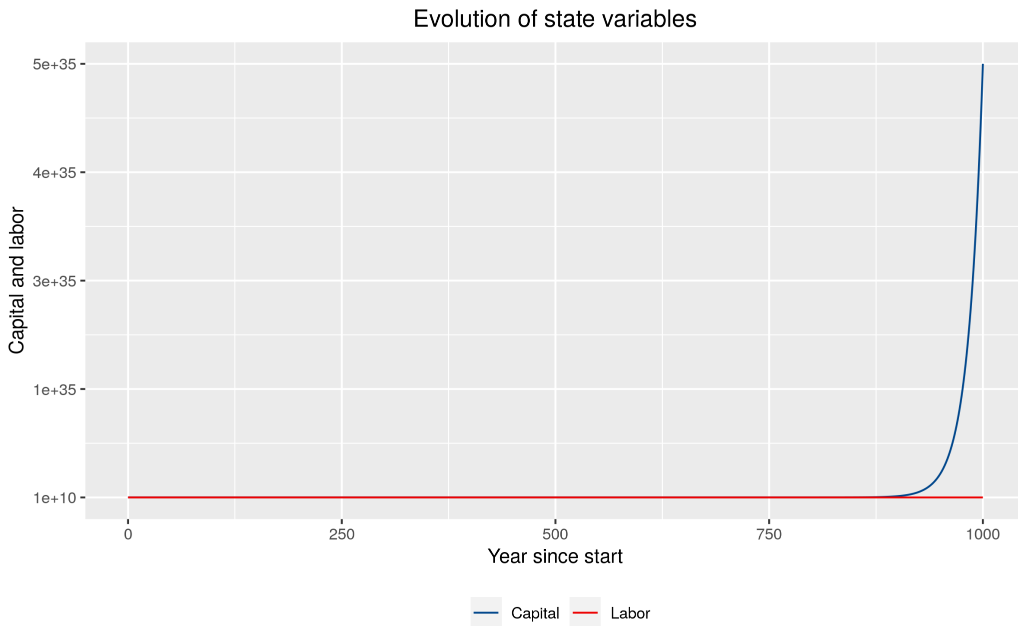
We can run some simulations to ellucidate the short-run behaviour under the model. Details of these simulations can be found in §B. More graphs can be found in §C. To do this, we will consider the variable values in our second example, (2.4.2), in addition to:

$$\begin{aligned} r_2 &= -0.05 \\ w_2 &= 2000 \\ \beta_2 &= 0.5 \\ x_1(t_0) &= 10^{10} \\ x_2(t_0) &= 10^5 \end{aligned} \tag{55}$$

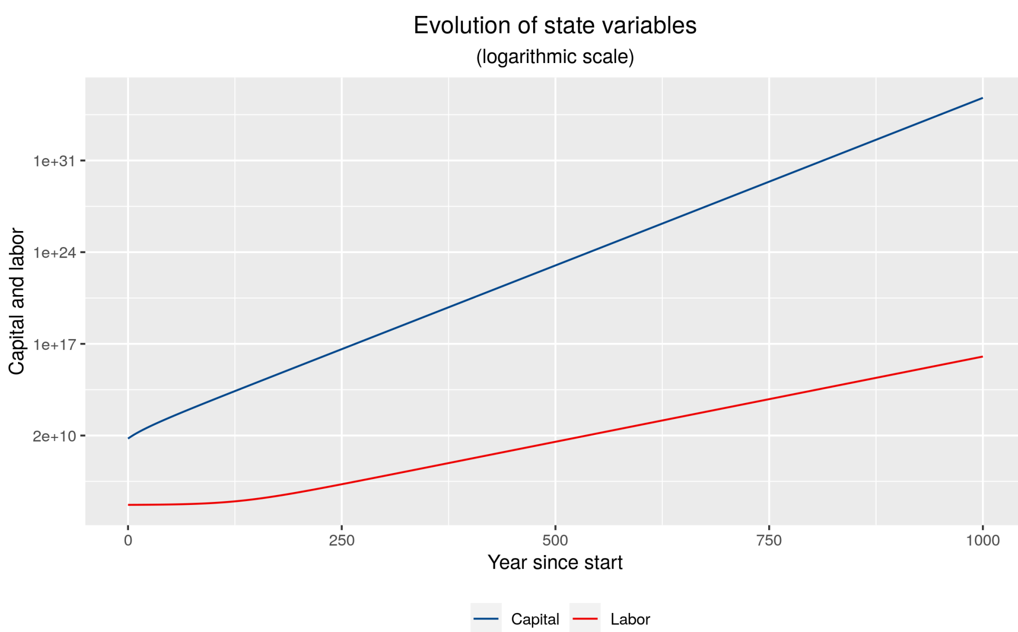
The r_2 value corresponds to a value drift (or death without replacement) rate of 5% per year. The w_2 value corresponds to a movement participant donating \$2000 per year, or 5% of a \$40.000 salary. The β_2 value corresponds to a team of five people being able to convince 5 other people a year on a 20k budget (and mantaining those they have convinced previously.) The initial values for x_1 and x_2 correspond to a \$10 billion endowment and 100k individuals broadly aligned with EA values. Further work could be done in order to determine more accurate and realistic estimates.

3.2 State variables

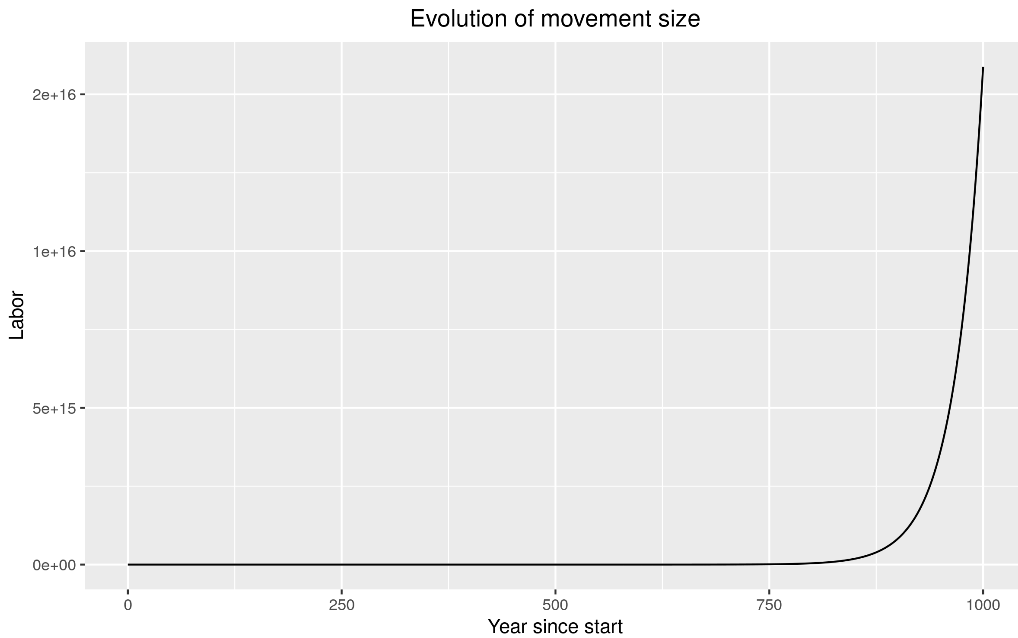
In this regime, the state variables, after an initial period in which labor stays roughly constant, these grow at an exponential rate:



Evolution of state variables
5e+35
4e+35
3e+35
1e+35
1e+10
Capital and labor
0
250
500
750
1000
Year since start
Capital
Labor

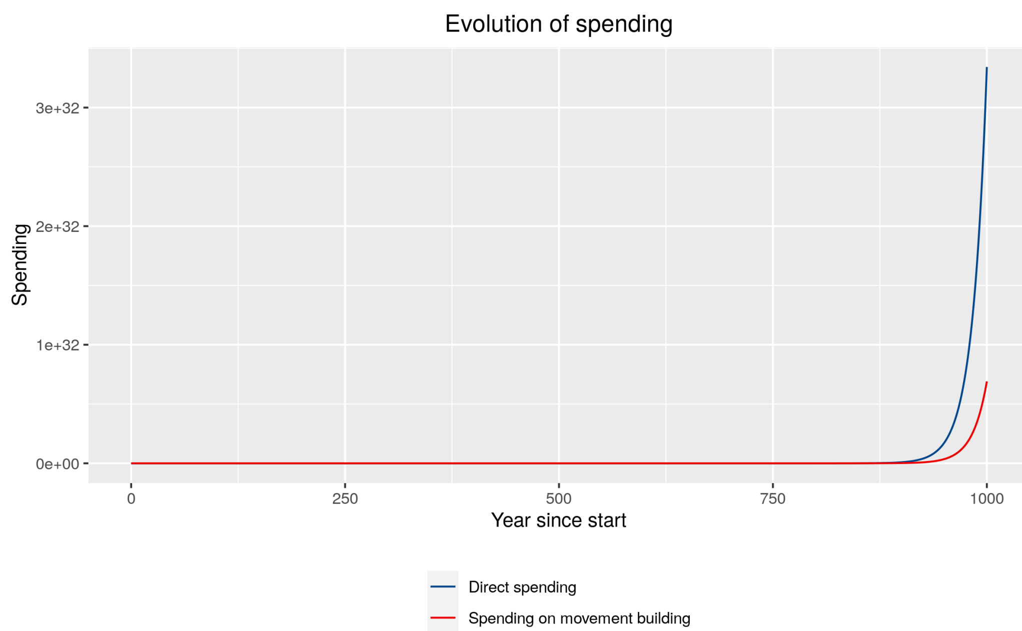


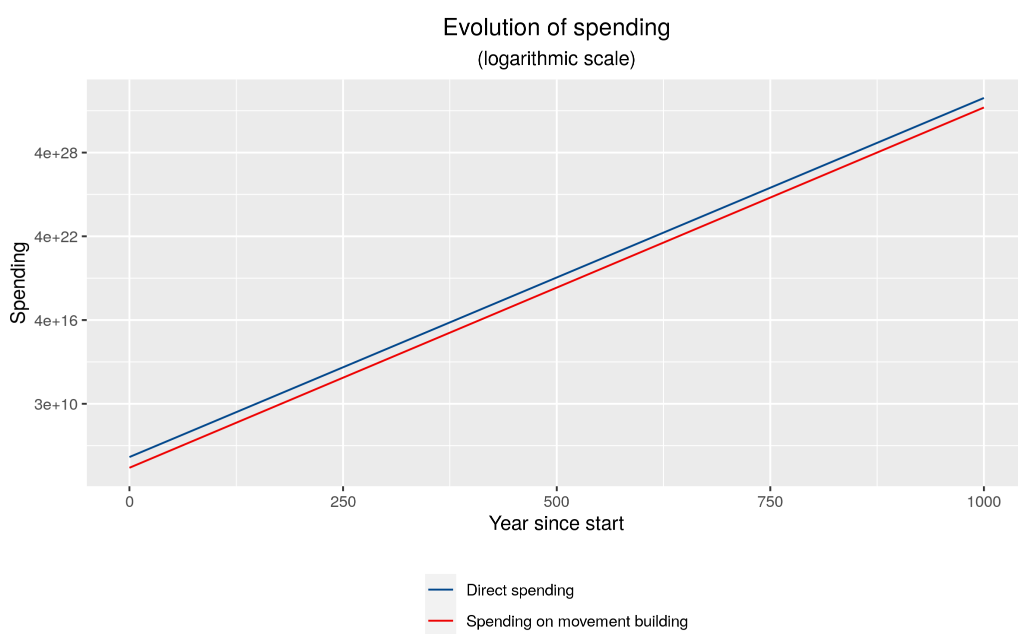
Evolution of state variables
(logarithmic scale)
1e+31
1e+24
1e+17
2e+10
Capital and labor
0
250
500
750
1000
Year since start
Capital
Labor



3.3 Spending rates

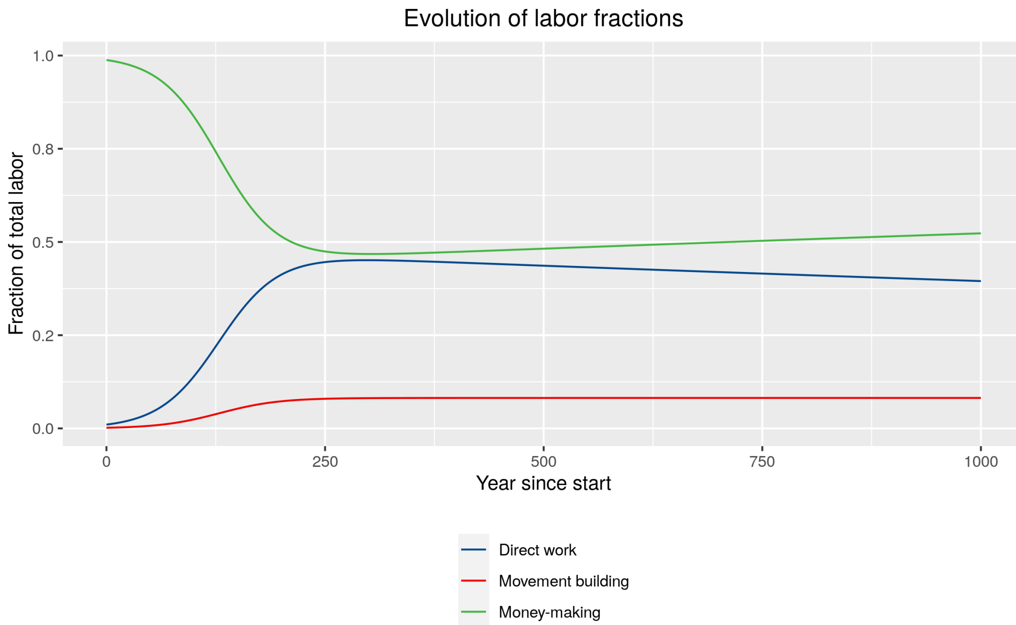
Spending also grows exponentially, as per results in (2.3.3). Note that, per (2.4.2), α_1 grows at a rate of 5.94%, whereas α_2 grows at a rate of 5.96%, so eventually, α_2 will catch-up with and surpass α_1. However, when it does so, the difference will be small enough to not be immediately apparent in a graph.


Evolution of spending
3e+32
2e+32
1e+32
0e+00
Spending
0
250
500
750
1000
Year since start
Direct spending
Spending on movement building


Evolution of spending
(logarithmic scale)
4e+28
4e+22
4e+16
3e+10
Spending
0
250
500
750
1000
Year since start
Direct spending
Spending on movement building

3.4 Allocation of labor

With regards to the allocations of labor, we observe the following:

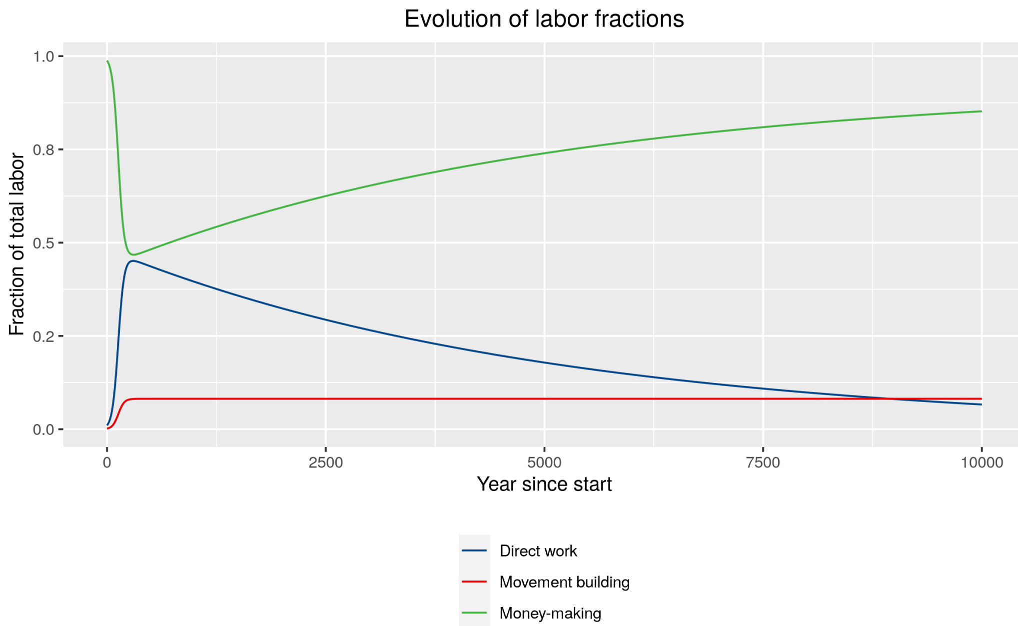


The starting point is a nearly 100% allocation of labor towards money-making. This is caused by our model assuming that wages grow more slowly than the return rate on capital.

For the purposes of illustration, consider a social movement made exclusively of airline pilots, and suppose that their wages had been steadily declining as their profession becomes commoditized. Then the optimal allocation is for them to accumulate money at the beginning, and then transition to direct work once their profession is paid less.[1] This example is imperfect because the example's tension is between pilots' salaries relative to other salaries, but in our graph and model the tension is between the donations of money-makers and the interest rate, yet the result is similar.

As time goes on, a different dynamic kicks in, and we observe:

[1]For another example, consider a social movement made up exclusively of Elon Musks, who have the ability to create valuable companies. Then the optimal path mgight involve the clones creating said companies and leaving philanthropy to their (due to regression to the mean, in expectation) less entrepeneurially competent descendants. This example might also apply to "Effective Altruism", which has a great proportion of members with a background in software engineering, which currently has a reputation for being a well-paying profession but might become more commoditized in the future.



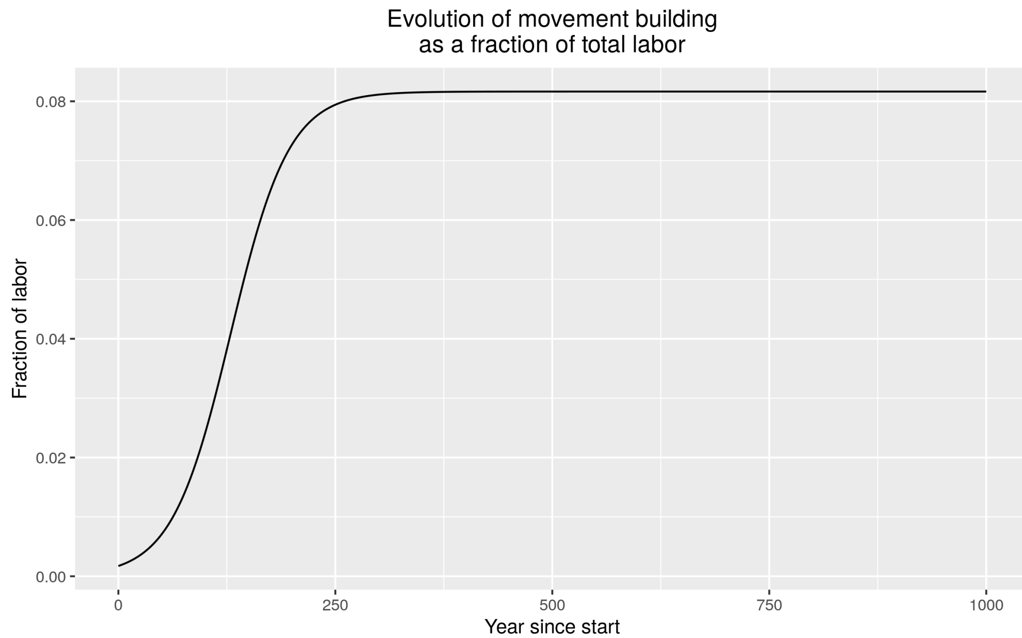
Recall the law of motion for x_2:

$$\dot{x}_2 = r_2 x_2 + \beta_2 \cdot exp\{\gamma_2 t\} \cdot (\alpha_2^{\lambda_2} \cdot (\sigma_2 x_2)^{1-\lambda_2})^{\delta_2} \tag{56}$$

Here, the best way to increase the absolute number of movement participants doing direct work turns out to be by investing into movement building. For any given growth rate in the absolute number of direct workers, g, the labor and capital inputs to movement-building term, $(\alpha_2^{\lambda_2} \cdot (\sigma_2 x_2)^{1-\lambda_2})^{\delta_2}$ has to grow at at least that rate (and a little bit more to adjust for the drift rate, r_2). In principle this could be accounted solely by a very fast growth rate on α_2, but in actuality, as α_2 grows, $x_2 \cdot \sigma_2$ would become the limiting factor, and σ_2 ends up converging to a constant, and we obtain our Quasi-Ponzi condition:

$$\frac{\text{direct workers}}{\text{direct workers + movement-builders}} \to 0$$

We conclude this section with a close-up view of σ_2:


Evolution of movement building
as a fraction of total labor
0.08
0.06
0.04
0.02
0.00
Fraction of labor
0
250
500
750
1000
Year since start

4 Conclusions

4.1 Outline of results

We have considered a stylized model of movement building in the context of social movements which aim to effect some change in the world.

In §2.2 we derived a necessary but not sufficient heuristic which might be used to check whether one is on the optimal path according to our model.

In §2.3, we derived the balanced growth rates for the stylized model, and in §2.3.3 we derived the spending path, that is, the optimal amount to spend on movement building at any given time.

We found that, for a space of plausible parameters, the optimal allocation implies an *asymptotic Ponzi* condition, where, even as the number of movement participants doing direct work grows with time in absolute terms, they converge to 0% of the total movement size, with most of the movement participants working either on earning money or in movement building. Analogously, even as the amounts of money spent on direct work grows in absolute terms, this amount also converges to 0% of total yearly spending, with most spending being directed towards movement building.

However, when carrying out numerical simulations, we find that this asymptotic Ponzi condition is indeed asymptotic, and doesn't instantiate itself in the immediate future.

When carrying out these simulations, we find that for some plausible parameters, the fraction of movement participants who do direct work grows until it reaches a peak, and then declines with time in favour of the fraction which dedicates themselves to earning money. Empirically, the exact magnitude and location of this peak depends heavily on the choice of k_1, a difficult to estimate constant, but the overall dynamic of growing and then declining doesn't.

4.2 Transversality violations

We also find that the problem under consideration displays a strong proclivity to violate the transversality conditions, that is, to generate seemingly impossible results. For example, if the amount of money and manpower needed to convince someone to join a social movement is and remains much lower than the amount of money and manpower which typical members are willing to give to this movement, and if these typical members are willing to allocate that money and manpower towards movement building, the optimal solution looks like an almost instantaneous recursive loop which quickly "takes over the world." This is the motivation for the $\delta_2 < 1$ term in (3).

Another type of transversality violation are the Satan's apple scenarios, such as those in (Arntzenius et al. 2003) [3]. In these kinds of scenarios, waiting $n + 1$ years might always be strictly better than waiting n years, but waiting forever is strictly worse than waiting any finite amount. In our case, this might correspond to a situation where investing for $n + 1$ years before spending is better than investing for only n years, but where investing forever and never spending is worse than investing for any finite amount of time. Similarly, it might be the case that directing all of a movement's resources and manpower towards movement building for n years to produce explosive movement growth, and then switching over to generating utility is only dominated by doing the same thing for $m > n$ years, but that solely concentrating on movement building forever would be suboptimal.

Now, for a range of plausible parameters this doesn't happen, but there is also no particular reason why one can't fall in a Satan's apple scenario. Arntzenius et al. argue that the rational choice in such a scenario is to stick to a large finite integer and to stop at that point.

4.3 Implicatons

In the short term, for our plausible parameters, our stylized model outputs that the optimal path involves mostly money-making, as opposed to either movement building or direct work.

In the long run, our stylized model allocates something of the order $\approx 10\%$ of labor and a majority of spending to movement building, which emphasizes its importance. Vast as Bill Gates' fortune may be, it is likely that most of his altruistic impact probably comes from the even greater billions which others have donated because of his Giving Pledge. On that note, spending numbers for the Giving Pledge are not readily available, and it is unclear what amount of effort it takes to persuade a billionaire to part with half of their fortune for philanthropic causes, but it might not be surprising if the Giving Pledge's budget was too low.

The timelines we consider, 1000 to 10000 years are such that for a social movement following the optimal path outlined in our stylized model, the aim should be to belong to the reference class of major religions and systems of thought, such as Zoroastrianism, Christianity or Confucianism.

As such, spending most of a social movement's capital within a generation, as Open Philanthropy, a major organization within the "Effective Altruism" ecosystem, intends to do, would in our model leave much utility on the table. Similarly, some aspiring effective altruists occasionally express the desire to "Keep EA weird"; our model suggests this is suboptimal.

4.4 Closing remarks

Overall, our results are contingent on the stylized movement building model capturing enough facets of reality to be of interest, but there are many respects in which it is not exhaustive. To mention two salient omissions, we don't consider global catastrophic or existential risks (such as runaway climate change, unaligned artificial intelligence, nuclear brinksmanship, extremely deadly global pandemics, etc.), which might lead us to consider more impatient allocations, and we also don't here consider the interplay between philanthropists who have different rates of time discounting.

Should it then the case that the stylized model is too far removed from reality, it may still serve as a building block for later and more detailed models which take into account these and further considerations. Indeed, current spending decisions seem to be the result of expert judgment calls rather than the result of optimal control theory calculations, and at this stage of modelling, expert intuition might provide better recommendations that those of our stylized model. Yet a research agenda aiming to model optimal allocations while taking into acccount all crucial considerations could be fleshed out and funded.

5 References

[1] Ramsey, F. P. (1928). "A Mathematical Theory of Saving". *The Economic Journal* Vol. 38, No. 152, pp. 543-559. Available at URL: http://piketty.pse.ens.fr/files/Ramsey1928.pdf. (archived URL). Last accessed September 24, 2020.

[2] Trammell, P. (2020). "Discounting for Patient Philanthropists" (Early draft). URL: https://philipTrammelll.com/static/discounting_for_patient_philanthropists.pdf (archived URL). Last accessed September 24, 2020.

[3] Arntzenius, F., Elga, A., & Hawthorne, J. (2004). "Bayesianism, Infinite Decisions, and Binding". Mind, 113(450), new series, pp. 251-283. URL: https://core.ac.uk/download/pdf/11920777.pdf (archived URL) Last accessed September 24, 2020.

[4] Kurlat, P. (2013). "Continuous Time Optimization, Part 1". URL: https://sites.google.com/view/pkurlat/teaching (archived URL). Last accessed September 24, 2020.

[5] Romer, P. (1986). "Cake Eating, Chattering, and Jumps: Existence Results for Variational Problems". Econometrica, 54(4), pp. 897-908. URL: https://www.jstor.org/stable/1912842 (archived URL). Last accessed September 24, 2020.

[6] Jones, C.I.. "Life and Growth". Journal of Political Economy, April 2016, Vol. 124 (2), pp. 539-578. URL: https://web.stanford.edu/~chadj/LifeandGrowthJPE2016.pdf (archived URL). Last accessed September 24, 2020.

[7] Aschenbrenner, L. (2020) "Existential Risk and Growth" (Early draft). URL: https://leopoldaschenbrenner.github.io/xriskandgrowth/ExistentialRiskAndGrowth050.pdf (archived URL). Last accessed September 24, 2020.

Appendices

A Proofs and derivations

A.1 Hamiltonian equations

$\frac{\partial H}{\partial \alpha_1} = 0$

$$(1-\eta)\cdot\lambda_1\cdot\frac{U}{\alpha_1}-\mu_1=0 \tag{57}$$

$$\mu_1=(1-\eta)\lambda_1\cdot\frac{U}{\alpha_1} \tag{58}$$

$\frac{\partial H}{\partial \alpha_2} = 0$

$$\mu_2\cdot\delta_2\lambda_2\cdot\frac{F_2}{\alpha_2}-\mu_1=0 \tag{59}$$

$$\mu_1=\mu_2\cdot\delta_2\cdot\lambda_2\cdot\frac{F_2}{\alpha_2} \tag{60}$$

$\frac{\partial H}{\partial \sigma_1} = 0$

$$(1-\eta)(1-\lambda_1)\cdot\frac{U}{\sigma_1}-\mu_1\cdot x_2\cdot w_2\cdot exp\{\gamma_1 t\}=0 \tag{61}$$

$$\mu_1=\frac{(1-\eta)(1-\lambda_1)}{w_2}\cdot\frac{U}{\sigma_1\cdot x_2\cdot exp\{\gamma_1 t\}} \tag{62}$$

$\frac{\partial H}{\partial \sigma_2} = 0$

$$-\mu_1\cdot x_2\cdot w_2\cdot exp\{\gamma_1 t\}+\mu_2\cdot\delta_2(1-\lambda_2)\cdot\frac{F_2}{\sigma_2}=0 \tag{63}$$

$$\mu_1=\mu_2\cdot\frac{\delta_2\cdot(1-\lambda_2)}{w_2}\cdot\frac{F_2}{\sigma_2\cdot x_2\cdot exp\{\gamma_1 t\}} \tag{64}$$

$\frac{\partial H}{\partial x_1} = \rho\mu_1-\dot{\mu}_1$

$$\mu_1\cdot r_1=\rho\mu_1-\dot{\mu}_1 \tag{65}$$

$$\mu_1=k_1\cdot exp\{(\rho-r_1)t\} \tag{66}$$

$\frac{\partial H}{\partial x_2} = \rho\mu_2 - \dot{\mu}_2$

$$\begin{aligned}\rho\mu_2 - \dot{\mu}_2 = \mu_2 \cdot (\rho - g_{\mu_2}) = (1-\eta) \cdot (1-\lambda_1) \cdot \frac{U}{x_2} \\ + \mu_1 \cdot w_2 \cdot exp\{\gamma_1 t\} \cdot (1 - \sigma_1 - \sigma_2) \\ + \mu_2 \cdot \left(r_2 + (1-\lambda_2) \cdot \delta_2 \cdot \frac{F_2}{x_2} \right)\end{aligned} \tag{67}$$

Through several manipulations of (67), in particular by substituting $(1 - \eta) \cdot (1 - \lambda_1) \cdot U$ from (62) and $(1 - \lambda_1) \cdot \delta_2 \cdot F_2 \cdot \mu_2$ from (64), we arrive at:

$$\mu_2 \cdot (\rho - g_{\mu_2} - r_2) = \mu_1 \cdot w_2 \cdot exp\{\gamma_1 t\} \tag{68}$$

This produces the growth equation

$$g_{\mu_1} = g_{\mu_2} + g_{x_2} - \gamma_1 \tag{69}$$

Summary

$$\mu_1 = (1-\eta)\lambda_1 \cdot \frac{U}{\alpha_1} \tag{70}$$

$$\mu_1 = \mu_2 \cdot \delta_2 \cdot \lambda_2 \cdot \frac{F_2}{\alpha_2} \tag{71}$$

$$\mu_1 = \frac{(1-\eta)(1-\lambda_1)}{w_2} \cdot \frac{U}{\sigma_1 \cdot exp\{\gamma_1 t\}} \tag{72}$$

$$\mu_1 = \mu_2 \cdot \frac{\delta_2 \cdot (1-\lambda_2)}{w_2} \cdot \frac{F_2}{\sigma_2 \cdot exp\{\gamma_1 t\}} \tag{73}$$

$$\mu_1 = k_1 \cdot exp\{(\rho - r_1)t\} \tag{74}$$

$$\mu_2 \cdot (\rho - g_{\mu_2} - r_2) = \mu_1 \cdot w_2 \cdot exp\{\gamma_1 t\} \tag{75}$$

A.2 Balanced growth equations

This last equation, (81) comes from $F_2 = \dot{x}_2 - r_2 x_2$. In the balanced growth path, $\dot{x}_2 = g_{x_2} \cdot x_2$, so $F_2 = g_{x_2} \cdot x_2 - r_2 x_2 = (g_{x_2} - x_2) \cdot x_2$.

$$g_{\mu_1} = g_U - g_{\alpha_1} \tag{76}$$

$$g_{\mu_1} = g_{\mu_2} + g_{F_2} - g_{\alpha_2} \tag{77}$$

$$g_{\mu_1} = g_U - g_{\sigma_1} - g_{x_2} - \gamma_1 \tag{78}$$

$$g_{\mu_1} = g_{\mu_2} + g_{F_2} - g_{\sigma_2} - g_{x_2} - \gamma_1 \tag{79}$$

$$g_{\mu_1} = (\rho - r_1) \tag{80}$$

$$g_{\mu_1} = g_{\mu_2} - \gamma_1 \tag{81}$$

$$g_{x_2} = g_{F_2} = \gamma_2 + \delta_2 \cdot \Big(\lambda_2 \cdot g_{\alpha_2} + (1 - \lambda_2) \cdot (g_{\sigma_2} + g_{x_2}) \Big) \tag{82}$$

Some simple simplifications follow. (88) is derived from (79) + (81) + $(g_{x_2} = g_{F_2})$.

$$g_{\alpha_1} = g_{\sigma_1} + g_{x_2} + \gamma_1 \tag{83}$$

$$g_{\mu_1} = g_U - g_{\alpha_1} \tag{84}$$

$$g_{\alpha_2} = \cancel{g_{\sigma_2}} + g_{x_2} + \gamma_1 \tag{85}$$

$$g_{\mu_1} = g_{\mu_2} + g_{F_2} - g_{\alpha_2} \tag{86}$$

$$g_{\mu_1} = \rho - r_1 \tag{87}$$

$$g_{\sigma_2} = 0 \tag{88}$$

$$g_{x_2} = g_{F_2} = \gamma_2 + \delta_2 \cdot \Big(\lambda_2 \cdot g_{\alpha_2} + (1 - \lambda_2) \cdot (g_{\sigma_2} + g_{x_2}) \Big) \tag{89}$$

A.3 Balanced growth path derivation

From this we can simply derive g_{x_2}, by substituting (85) and (88) in (89)

$$g_{x_2} = \frac{\gamma_2 + \delta_2 \lambda_2 \gamma_1}{1 - \delta_2} \tag{90}$$

And from that g_{α_2}, by substituting (90) back in (85)

$$g_{\alpha_2} = g_{x_2} + \gamma_1 = \frac{\gamma_2 + \delta_2 \lambda_2 \gamma_1}{1 - \delta_2} + \gamma_1 \tag{91}$$

Similarly, from (83), (84) and (90), we can derive g_{α_1} and g_{σ_1}:

$$g_{\alpha_1} = \frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta} \cdot \gamma_1 \tag{92}$$

$$g_{\sigma_1} = \frac{r - \rho}{\eta} - \left(\frac{(1-\eta)(1-\lambda_1)}{\eta} + 1 \right) \cdot \gamma_1 - g_{x_2} \tag{93}$$

Note that this solution is only valid where $g_{\sigma_1} \leq 0$.

Note also that $g_{\alpha_1} \leq g_{\alpha_2}$. Proof: $g_{\alpha_1} = g_{\sigma_1} + g_{x_2} + \gamma_1$, and $g_{\alpha_2} = \cancel{g_{\sigma_2}} + g_{x_2} + \gamma_1$. Hence $g_{\alpha_1} = g_{\sigma_1} + g_{\alpha_2} \wedge g_{\sigma_1} \leq 0 \implies g_{\alpha_1} \leq g_{\alpha_2}$.

We can also derive x_1.

$$\dot{x}_1 = r_1 x_1 - \alpha_1 - \alpha_2 + x_2 \cdot w_2 \cdot exp\{\gamma_1 t\} \cdot (1 - \sigma_1 - \sigma_2) \tag{94}$$

$$x_1 = a \cdot exp\{r_1 \cdot t\} + b \cdot exp\{g_{\alpha_1} \cdot t\} + c \cdot exp\{g_{\alpha_2} \cdot t\} \tag{95}$$

A.4 Checking the transversality condition

The variables we need follow. We get μ_2 from (79) + $(g_{F_2} = g_{x_2})$ + $(g_{\sigma_2} = 0)$

$$\mu_1 = k_1 \cdot exp\{(\rho - r_1) \cdot t\} \tag{96}$$

$$\mu_2 = k_2 \cdot exp\left\{ \Big((\rho - r_1) + \gamma_1 \Big) \cdot t \right\} \tag{97}$$

$$x_1 = a \cdot exp\{r_1 \cdot t\} + b \cdot exp\{g_{\alpha_1} \cdot t\} + c \cdot exp\{g_{\alpha_2} \cdot t\} \tag{98}$$

$$x_2 = exp\left\{\frac{\gamma_2 + \delta_2\lambda_2\gamma_1}{1-\delta_2}\cdot t\right\} \tag{99}$$

The transversality condition is

$$\lim_{t\to\infty} exp\{-\rho\cdot t\}\cdot x_i\cdot \mu_i = 0 \tag{100}$$

For $i = 1$, this implies $a = 0$, $g_{\alpha_1} < r_1$, $g_{\alpha_2} < r_1$. For $\rho \approx 0.005, \gamma_1 \approx 0.05, \gamma_2 \approx 0.01, r_1 \approx 0.06, \lambda_1 \approx 0.5$, this implies $\eta \gtrsim 0.86$.

For $i = 3$, the transversality condition is satisfied when:

$$-\rho + (\rho - r_1 + \gamma_1) + \frac{\gamma_2 + \delta_2\cdot\lambda_2\cdot\gamma_1}{1-\delta_2} < 0 \tag{101}$$

i.e.,

$$\gamma_1 + \frac{\gamma_2 + \delta_2\cdot\lambda_2\cdot\gamma_1}{1-\delta_2} < r_1 \tag{102}$$

or, alternatively,

$$g_{\alpha_2} = gx_1 + \gamma_1 < r_1 \tag{103}$$

For $\lambda_2 \approx 0.5, r_1 \approx 0.06, \gamma_1 \approx 0.03, \gamma_2 \approx 0.01$, this implies that either $\delta_2 \lesssim 0.44$ or $1 < \delta_2$. For $\gamma_1 \approx 0.02$, this changes to $-1 < \delta_2 \lesssim 0.6$ or $1 < \delta_2$.

Further, (68) implies $\rho - g_{x_2} - r_2 > 0$, i.e., $r_2 + \gamma_1 < r_1$; otherwise the first term in the equality in (68) would be negative and the second one positive. We will see in (119) that r_2, the movement drift rate, increases the initial value of α_2, but not its growth rate. Still, $r_2 + \gamma_1 < r_1$ allows only for a pretty low drift rate.

A.5 Exact spending schedules

In this section, through the previous equations, we derive a more or less explicit formula for α_1 and α_2. Using that, determine the form of σ_1 and σ_2, and having these, we derive the instantaneous change in x_1 and x_2, and this is already enough for numerical simulations.

To derive α_1, we will make use of the following equations: (70), (74) and (9)

$$\mu_1 = (1-\eta)\lambda_1 \cdot \frac{U}{\alpha_1} \tag{104}$$

$$\mu_1 = k_1 \cdot exp\{(\rho - r_1)t\} \tag{105}$$

$$\frac{\lambda_1}{\alpha_1} = \frac{1-\lambda_1}{\sigma_1 \cdot x_2 \cdot w_2 \cdot exp\{\gamma_1 t\}} \tag{106}$$

Expanding the full form of U per (2) on (104):

$$\mu_1 = \lambda_1 \cdot \frac{(\alpha_1^{\lambda_1}(\sigma_1 x_2)^{1-\lambda_1})^{1-\eta}}{\alpha_1} \tag{107}$$

and replacing $\sigma_1 \cdot x_2$ on (106) from (105):

$$\mu_1 = \lambda_1 \cdot \frac{\left(\alpha_1^{\lambda_1} \cdot \left(\frac{1-\lambda_1}{\lambda_1} \cdot \frac{\alpha_1}{w_2 \cdot exp\{\gamma_1 t\}}\right)^{1-\lambda_1}\right)^{1-\eta}}{\alpha_1} \tag{108}$$

$$\mu_1 = \lambda_1 \cdot \frac{\alpha_1^{(1-\eta)}}{\alpha_1} \cdot \left(\frac{1-\lambda_1}{\lambda_1 \cdot w_2 \cdot exp\{\gamma_1 t\}}\right)^{(1-\lambda_1)(1-\eta)} \tag{109}$$

$$\alpha_1^{\eta} = \frac{\lambda_1}{\mu_1} \cdot \left(\frac{1-\lambda_1}{\lambda_1 \cdot w_2 \cdot exp\{\gamma_1 t\}}\right)^{(1-\lambda_1)(1-\eta)} \tag{110}$$

$$\alpha_1^{\eta} = \frac{\lambda_1}{k_1 \cdot exp\{(\rho - r_1)t\}} \cdot \left(\frac{1-\lambda_1}{\lambda_1 \cdot w_2 \cdot exp\{\gamma_1 t\}}\right)^{(1-\lambda_1)(1-\eta)} \tag{111}$$

Note how this is consistent with (92):

$$g_{\alpha_1} = \frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta} \cdot \gamma_1 \tag{112}$$

Now, if k_1 is too small, then α_1 becomes so large that $x_1 \to -\infty$. Conversely, if k_1 is too large, then α_1 is too small and we accumulate money we are never to spend. k_1 will be then uniquely determined by being the value such that neither of those conditions hold.

We can derive α_2 in a similar manner, starting from (71), (10) and (75)

$$\mu_1 = \mu_2 \cdot \delta_2 \cdot \lambda_2 \cdot \frac{F_2}{\alpha_2} \tag{113}$$

$$\frac{\lambda_2}{\alpha_2} = \frac{1 - \lambda_2}{\sigma_2 \cdot x_2 \cdot w_2 \cdot exp\{\gamma_1 t\}} \tag{114}$$

$$\mu_2 \cdot (\rho - g_{\mu_2} - r_2) = \mu_1 \cdot w_2 \cdot exp\{\gamma_1 t\} \tag{115}$$

We expand F_2 on (113) per (3) and divide by μ_2:

$$\frac{\mu_1}{\mu_2} = \delta_2 \cdot \lambda_2 \cdot \frac{\beta_2 \cdot exp\{\gamma_2 t\} \cdot (\alpha_2^{\lambda_2} \cdot (\sigma_2 x_2)^{1-\lambda_2})^{\delta_2}}{\alpha_2} \tag{116}$$

We simplify μ_1/μ_2 per (115), replace $\sigma_1 x_2$ per (114), and substitute $g_{\mu_2} = \rho - r_1 + \gamma_1$

$$\frac{\rho - g_{\mu_2} - r_2}{w_2 \cdot exp\{\gamma_1 \cdot t\}} = \delta_2 \cdot \lambda_2 \cdot \beta_2 \cdot exp\{\gamma_2 t\} \cdot \frac{\left(\alpha_2^{\lambda_2} \cdot \left(\frac{1-\lambda_2}{\lambda_2} \cdot \frac{\alpha_2}{w_2 \cdot exp\{\gamma_1 t\}}\right)^{1-\lambda_2}\right)^{\delta_2}}{\alpha_2} \tag{117}$$

$$\frac{\rho - (\rho - r_1 + \gamma_1) - r_2}{w_2 \cdot exp\{\gamma_1 \cdot t\}} = \delta_2 \cdot \lambda_2 \cdot \beta_2 \cdot exp\{\gamma_2 t\} \cdot \frac{\alpha_2^{\delta_2}}{\alpha_2} \cdot \left(\frac{1-\lambda_2}{\lambda_2 \cdot w_2 \cdot exp\{\gamma_1 t\}} \cdot\right)^{\delta_2 \cdot (1-\lambda_2)} \tag{118}$$

$$\alpha_2^{1-\delta_2} = \frac{w_2 \cdot exp\{\gamma_1 \cdot t\}}{r_1 - \gamma_1 - r_2} \cdot \delta_2 \cdot \lambda_2 \cdot \beta_2 \cdot exp\{\gamma_2 t\} \cdot \left(\frac{1-\lambda_2}{\lambda_2 \cdot w_2 \cdot exp\{\gamma_1 t\}} \cdot\right)^{\delta_2 \cdot (1-\lambda_2)} \tag{119}$$

B Numerical simulation details

B.1 Overview

We have determined the value of α_i at all times (up to a constant k_1), as well as α_2. Now suppose we knew x_1 and x_2 at some point, for example at the present time t_0, i.e., $x_1(t_0), x_2(t_0)$. Then, we could also figure out $\sigma_i(t_0)$, per (9) and (10):

$$\sigma_i(t_0) = \frac{1-\lambda_i}{\lambda_i} \cdot \frac{\alpha_i(t_0)}{x_2(t_0) \cdot w_2 \cdot exp\{\gamma_1 t_0\}} \tag{120}$$

Using $\alpha_1(t_0), \alpha_2(t_0), \sigma_1(t_0), \sigma_2(t_0), x_1(t_0), x_2(t_0)$ we can approximate the derivative, or instantaneous change of the state variables, $\dot{x}_1(t_0), \dot{x}_2(t_0)$ per their law of motion (3), and then approximate $x_i(t_0 \pm \epsilon) = x_i(t_0) \pm \epsilon \cdot \dot{x}_i(t_0)$. Our general approach to generate numerical approximations will be to use this approximation.

The method in which we start with the values at some initial point in time and then extrapolate them into the future is known as forward shooting. In contrast, the method in which we try to guess some final points in the future which, when extrapolated into the past hit our initial conditions is known as reverse shooting. Reverse shooting is known for being more stable, but in this instance it fails, perhaps because of floating point errors.

The code, in R, is based on previous Matlab code originally by Charles Jones, modified by Leopold Aschenbrenner and cleaned up by myself. Aschenbrenner's code can be found in this online repository: GitHub.com/NunoSempere/ ReverseShooting, and my own code can be found in GitHub.com/NunoSempere/ Movement Building For Utility Maximizers, which contains more details about how to run it.

This code makes use of the variable values from our second example scenario in (2.4.2)

$$\begin{aligned} \eta &= 0.9 \\ \rho &= 0.005 \\ r_1 &= 0.06 \\ \gamma_1 &= 0.03 \\ \gamma_2 &= 0.01 \\ \lambda_1 &= 0.5 \\ \lambda_2 &= 0.5 \\ \delta_2 &= 0.44 \end{aligned} \tag{121}$$

To which we add r_2, which is negative because it represents a value-drift or drop-out rate (as opposed to, say, a fertility rate).

$$r_2 = -0.05 \tag{122}$$

and β_2, w_2.

$$\begin{aligned} w_2 &= 2000 \\ \beta_2 &= 0.5 \end{aligned} \tag{123}$$

These factors correspond to each movement participant donating \$2000 per year, or 5% of a \$40.000 salary, and a team of five people being able to convince 5 other people a year on a 20k budget (and mantaining those they have convinced previously.) Further work could be done in order to determine more accurate and realistic estimates. We also consider initial conditions:

$$x_1(t_0) = \texttt{x\_1\_init} = 10^{10} \tag{124}$$

$$x_2(t_0) = \texttt{x\_2\_init} = 10^{5} \tag{125}$$

We also consider two parameters, corresponding to our unknown constant k_1: `k1_forward_shooting` and `k1_reverse_shooting`. They determine spending on direct work. Their value is such that decreasing it results in too little spending, and the movement accumulates money which is never spent. Conversely, increasing it results in the movement going bankrupt and acquiring infinite debt. However, its value is inexact, and will be a source of error. In particular, if we run simulations until time t, we don't know that the movement will not go bankrupt at some subsequent time, and hence k_1 requires some guesswork. More specifically, if we select the maximum k_1 such that x_1 is positive at time t, we tend to find that $x_1 \to -\infty$ shortly afterwards.

```
k1_forward_shooting = 3*10^(-7)
k1_reverse_shooting = 3*10^(-7)
```

Finally, we decide on a step-size and on a time interval. The time interval will start at 100 years, and increase to 1,000 and then 10,000 years.

```
stepsize = 0.1
```

```
first = 0
last = 100
times_forward_shooting = seq(from=first, to=last, by=stepsize)
times_reverse_shooting = seq(from=last, to=first, by=-stepsize)
```

B.2 Problematic details

B.2.1 Floating point errors

Using a very small step size runs into floating point errors. Consider a stylized example:

```
options(digits=22)
dx <- 10^43
numsteps <- 10^7
stepsize <- 10^(-3)

## Example 1
x <- pi*1e+60
print(x)
for(i in c(1:numsteps)){
x <- x+dx*stepsize
}
x == pi*1e+60
# [1] TRUE

## Example 2
x <- pi*1e+60 + numsteps*stepsize*dx
x == pi*1e+60
# [1] FALSE
```

The two examples should give the same results, but don't.

B.2.2 Transversality violations

If the ratio between money and movement size is too large, a boundary condition violation can occur where $\sigma_1 > 1, \sigma_3 < 0$. The interpretation here is that σ_3 is negative because we choose to hire people at a rate of $w_3 \cdot exp\{\gamma_1 \cdot t\}$

B.2.3 Reverse shooting

Perhaps because of floating point errors, reverse shooting fails. Consider an stylized example

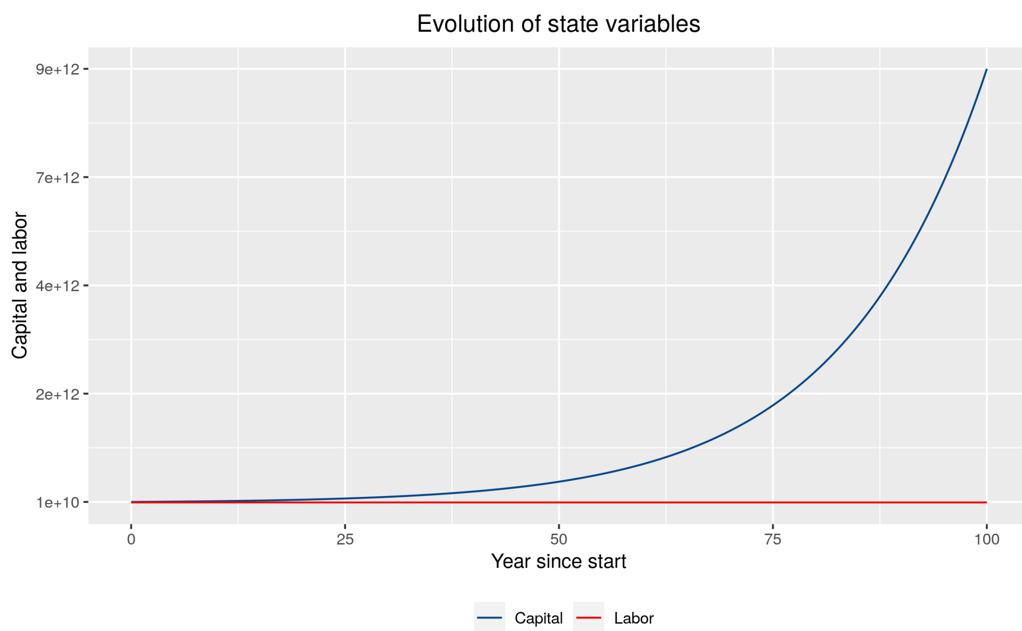
```
## Stylized forward shooting

x <- 0
for(i in c(1:30)){
  x <- x + 7^i
}

## Stylized reverse shooting

y <- x
for(i in c(30:1)){
  y <- y - 7^i
}
print(y)
# [1] -1227701488
```

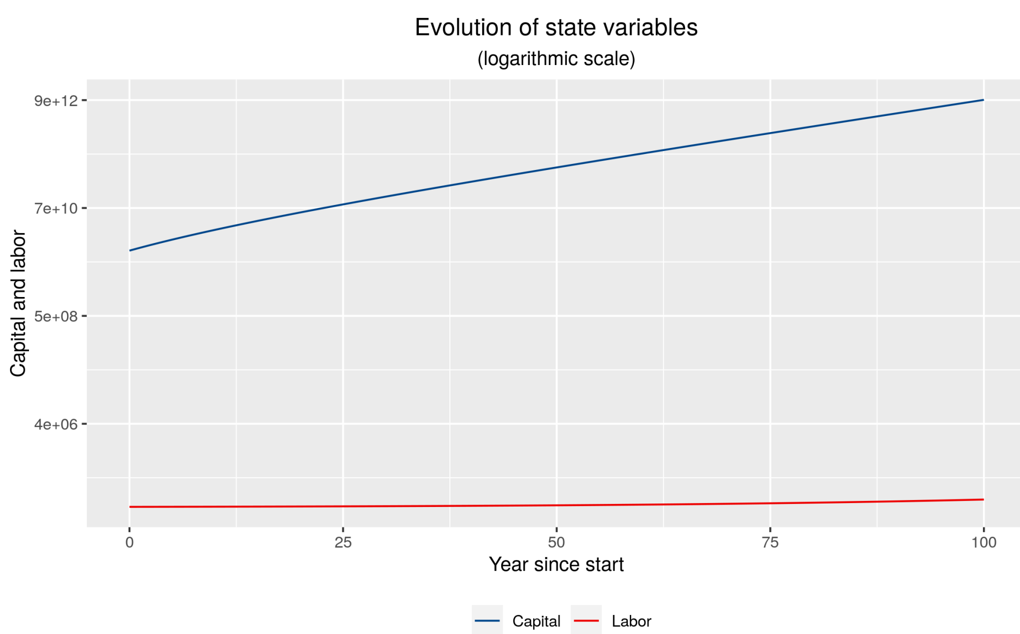
Here, `y` should at the end be 0, but floating point errors ensure that it isn't. Given that our variables grow exponentially, we work with very large numbers and reverse shooting encounters similar errors and fails. Hence, we are restricted to using forward shooting.

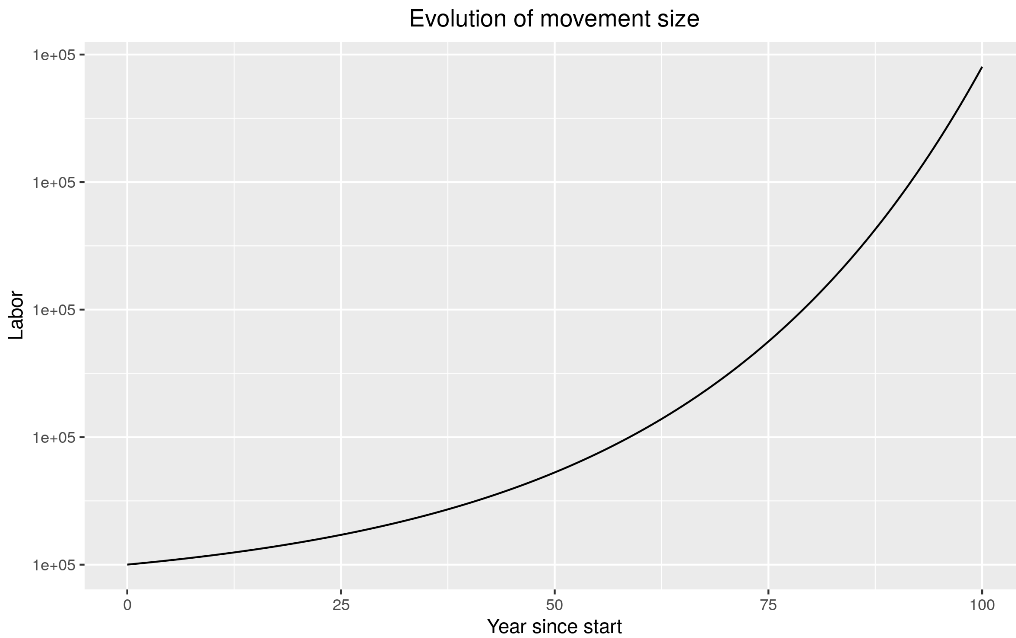
C Additional graphs

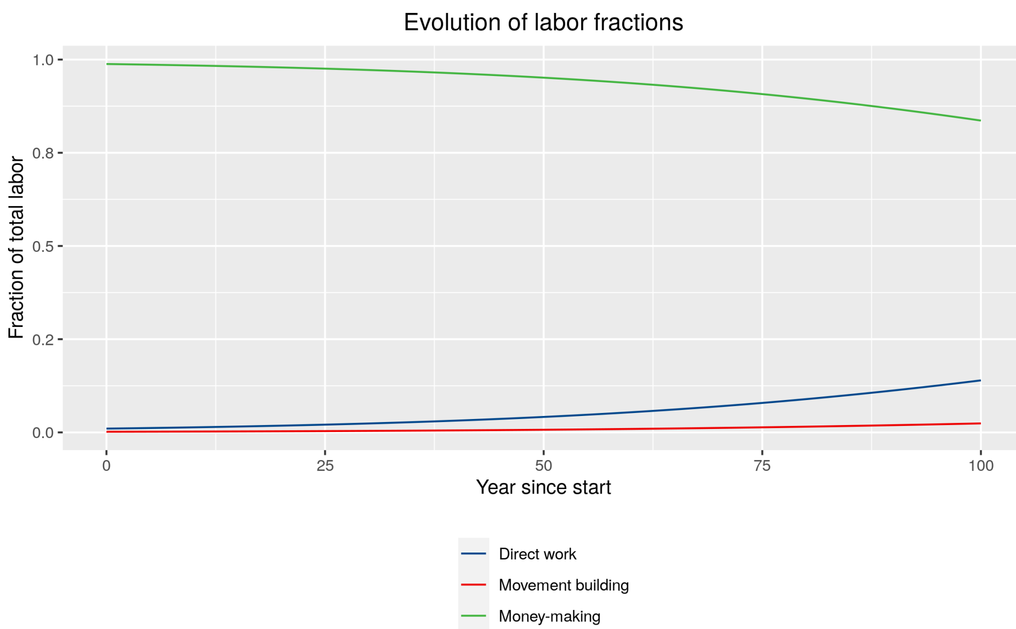
C.1 Graphical results: 100 years

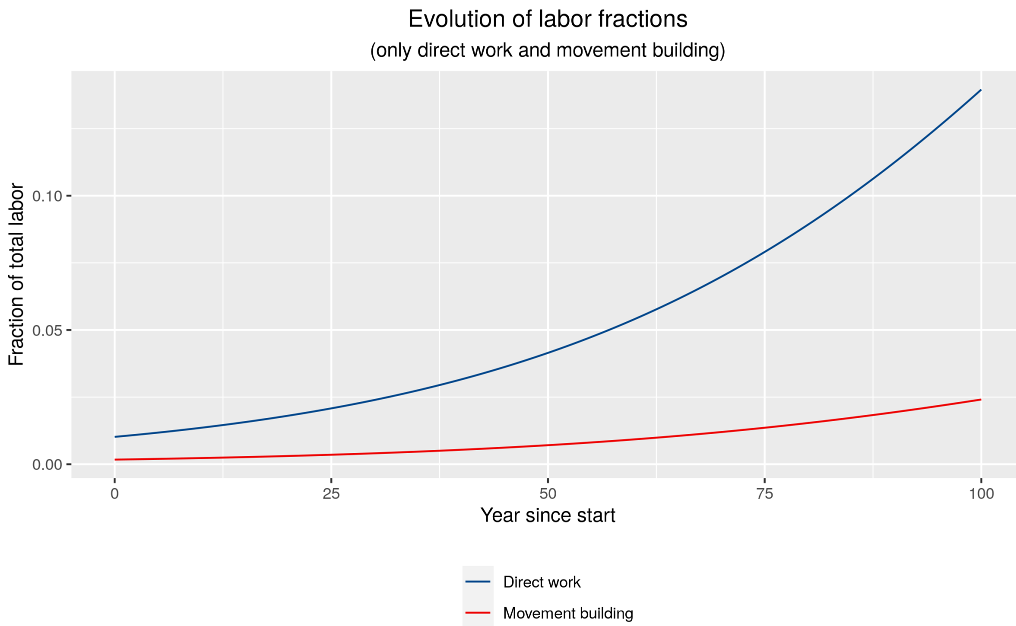
For the first hundred years, accumulated money and movement size grow at different exponential rates. The allocation of participants is primarily to money-making, though both the allocations of movement participants to direct work and to movement building initially increase exponentially, with the former doing so at a much higher rate. Spending also increases in absolute terms for both direct work and movement building (per (2.3.3)).

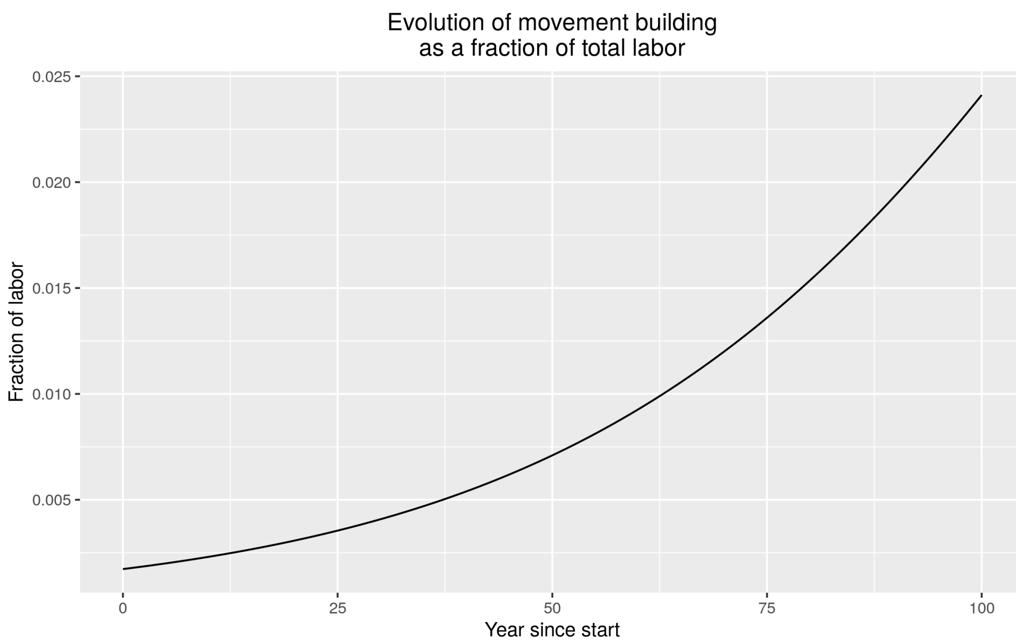

Evolution of state variables
9e+12
7e+12
4e+12
2e+12
1e+10
Capital and labor
0
25
50
75
100
Year since start
Capital
Labor

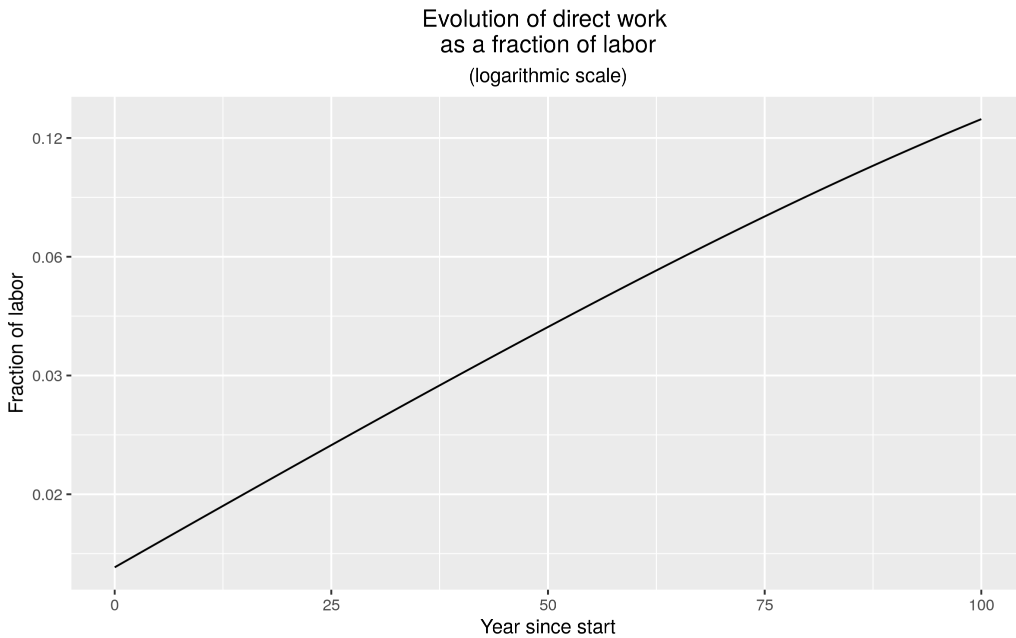

Evolution of state variables
(logarithmic scale)
9e+12
7e+10
5e+08
4e+06
Capital and labor
0
25
50
75
100
Year since start
Capital
Labor


Evolution of movement size
Labor
1e+05
1e+05
1e+05
1e+05
1e+05
0
25
50
75
100
Year since start

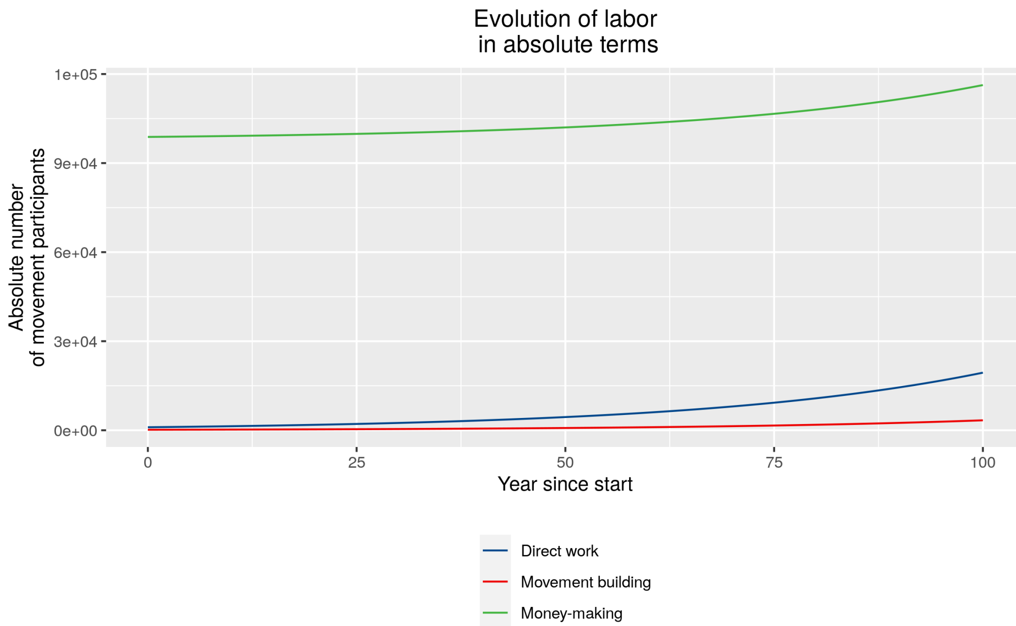

Evolution of labor fractions
Fraction of total labor
1.0
0.8
0.5
0.2
0.0
0
25
50
75
100
Year since start
Direct work
Movement building
Money-making

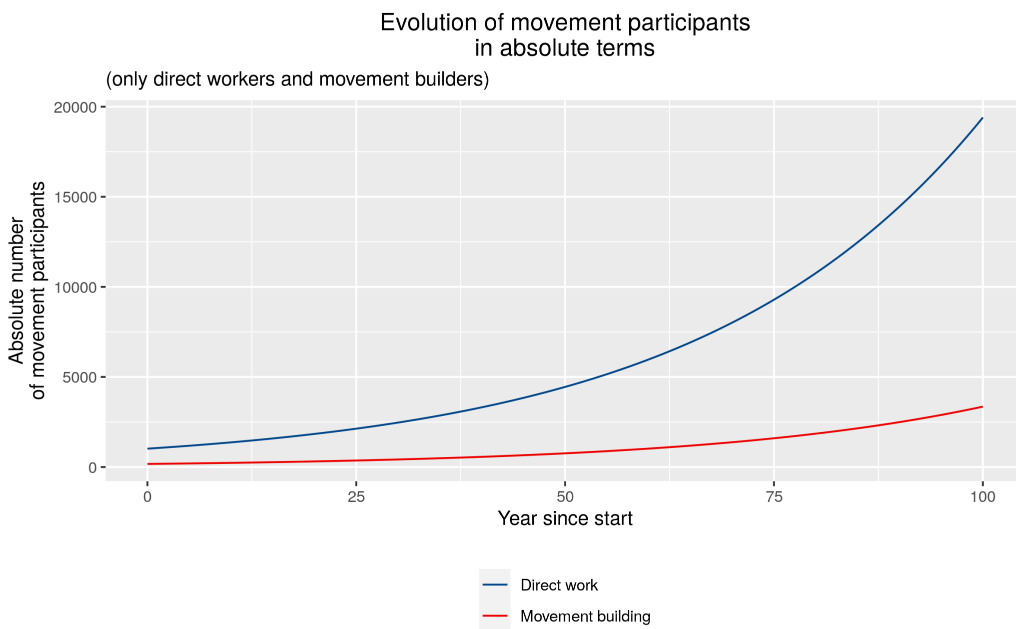

Evolution of labor fractions
(only direct work and movement building)
Fraction of total labor
0.10
0.05
0.00
0
25
50
75
100
Year since start
Direct work
Movement building


Evolution of movement building
as a fraction of total labor
Fraction of labor
0.025
0.020
0.015
0.010
0.005
0
25
50
75
100
Year since start

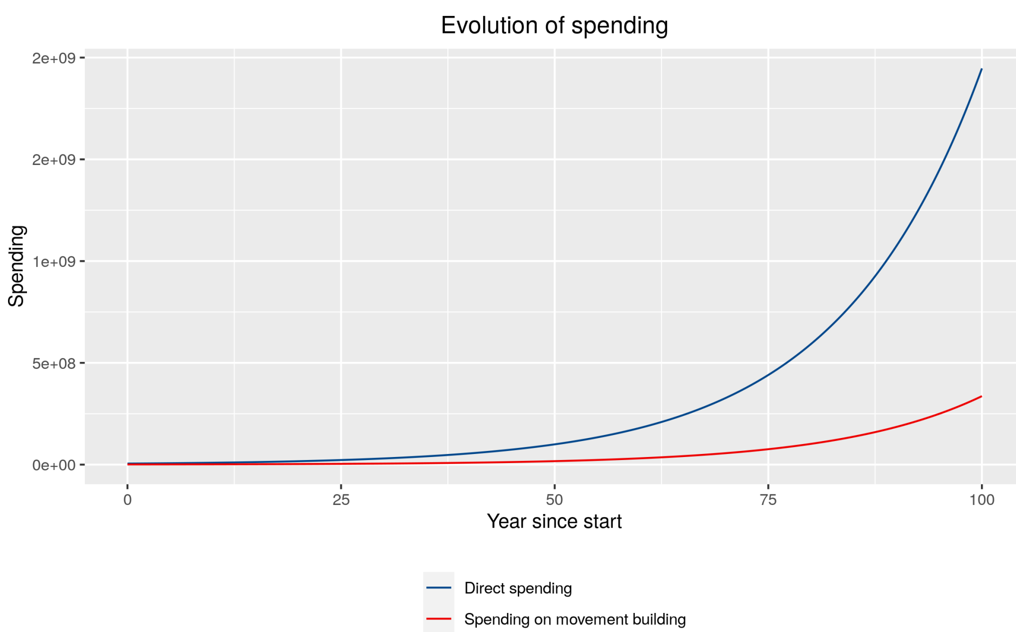

Evolution of direct work
as a fraction of labor
(logarithmic scale)
Fraction of labor
0.12
0.06
0.03
0.02
0
25
50
75
100
Year since start

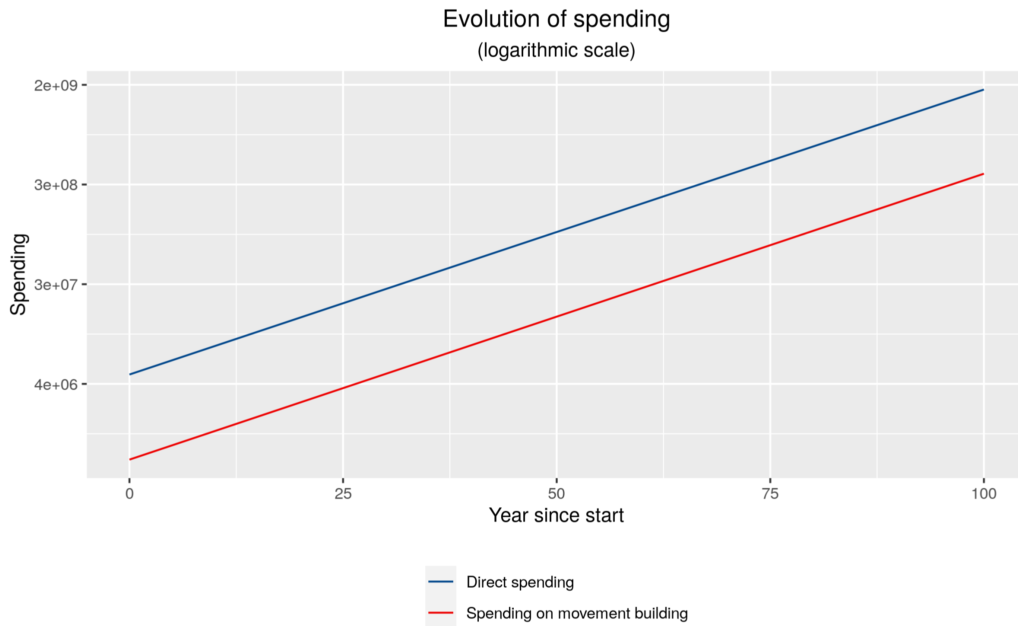

Evolution of movement building
as a fraction of labor
(logarithmic scale)
Fraction of total labor
0.016
0.008
0.004
0.002
0
25
50
75
100
Year since start


Evolution of labor
in absolute terms
Absolute number
of movement participants
1e+05
9e+04
6e+04
3e+04
0e+00
0
25
50
75
100
Year since start
Direct work
Movement building
Money-making


Evolution of movement participants
in absolute terms
(only direct workers and movement builders)
Absolute number
of movement participants
20000
15000
10000
5000
0
0
25
50
75
100
Year since start
Direct work
Movement building

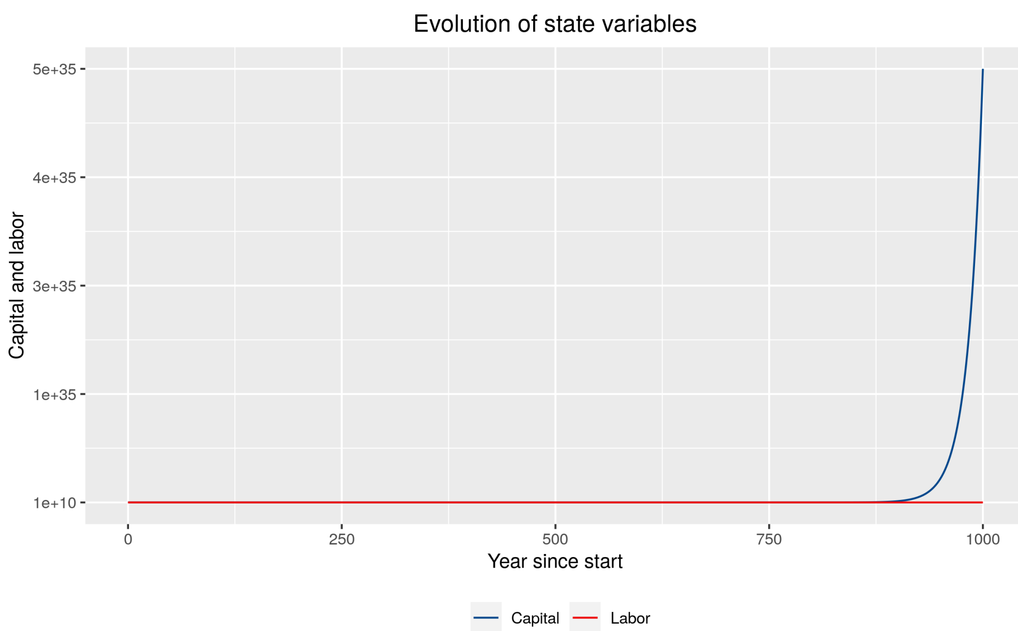

Evolution of movement participants
in absolute terms
(logarithmic scale)
(only direct workers and movement builders)
Absolute number
of movement participants
8192
2048
512
0
25
50
75
100
Year since start
Direct work
Movement building

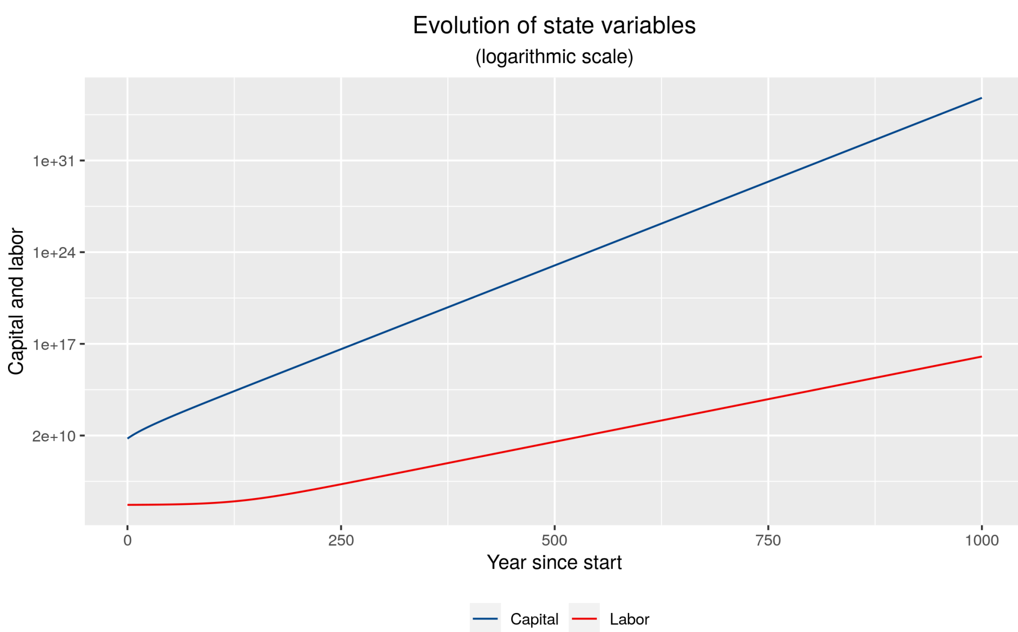

Evolution of spending
2e+09
2e+09
1e+09
5e+08
0e+00
Spending
0
25
50
75
100
Year since start
Direct spending
Spending on movement building

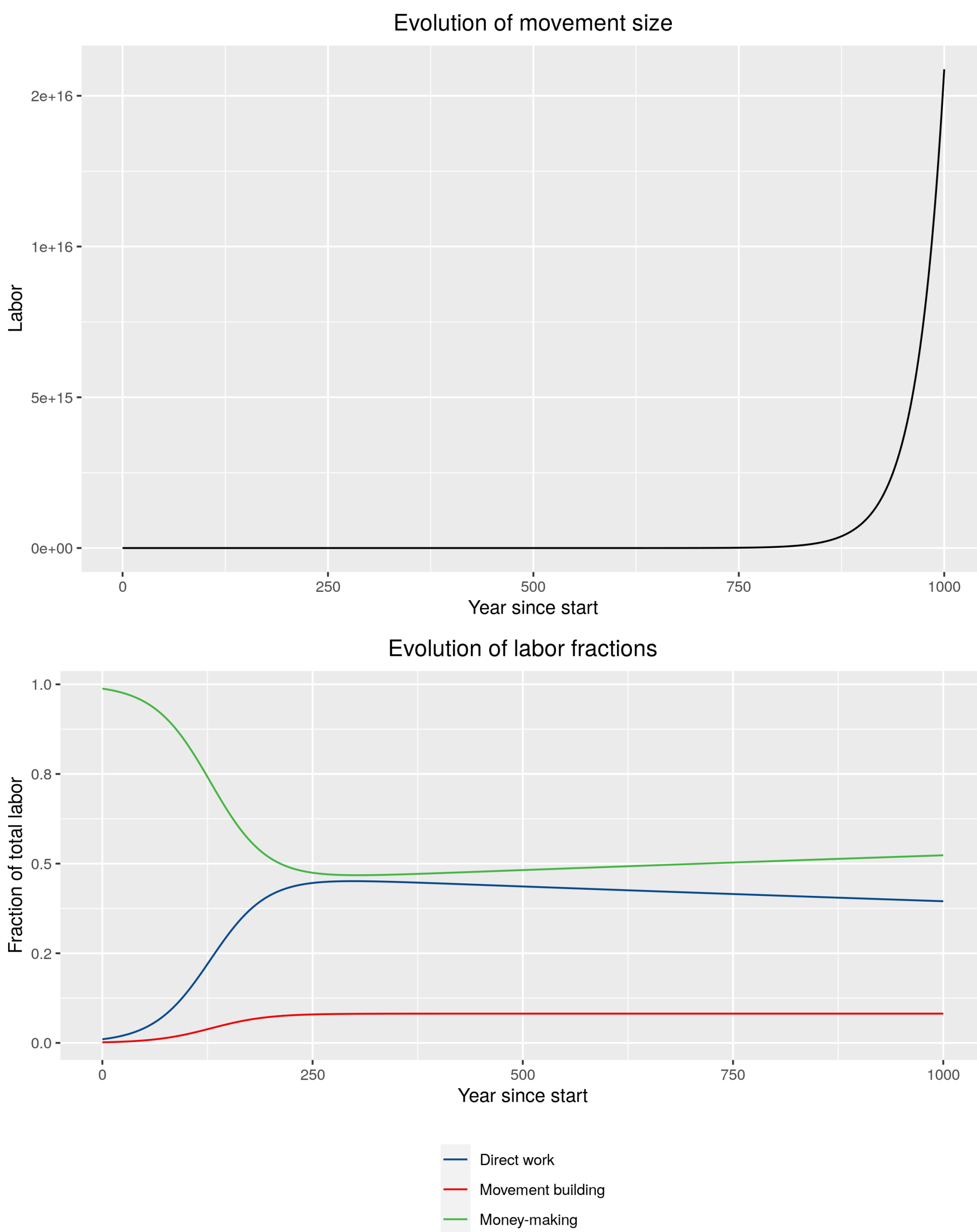


C.2 Graphical results: 1,000 years

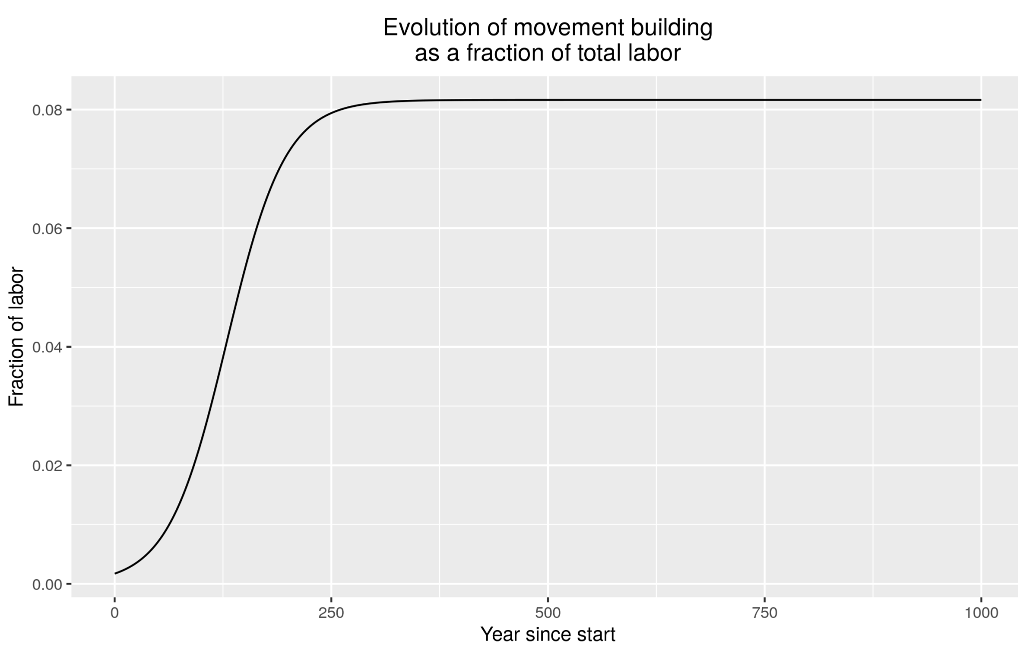
The dynamic for the state and spending variables is mostly as in the previous section. With regards to movement size and distribution, movement building as a fraction of movement size plateaus at around 0.65%, and stays there. Direct work reaches 40%, and starts slowly declining, whereas money-making starts increasing back-up once again.

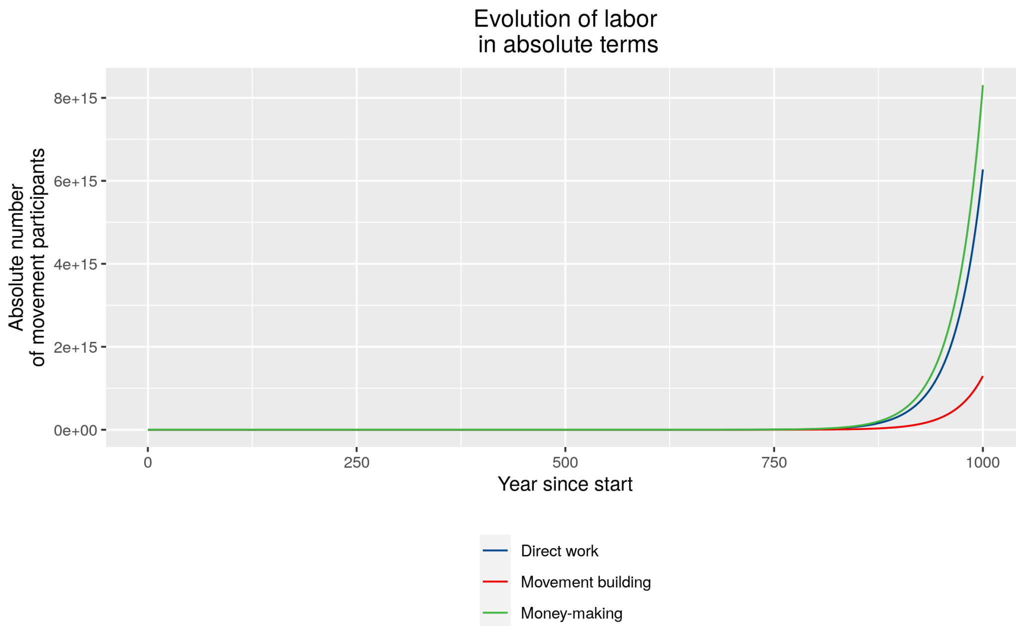

Evolution of state variables
5e+35
4e+35
3e+35
1e+35
1e+10
Capital and labor
0
250
500
750
1000
Year since start
Capital
Labor

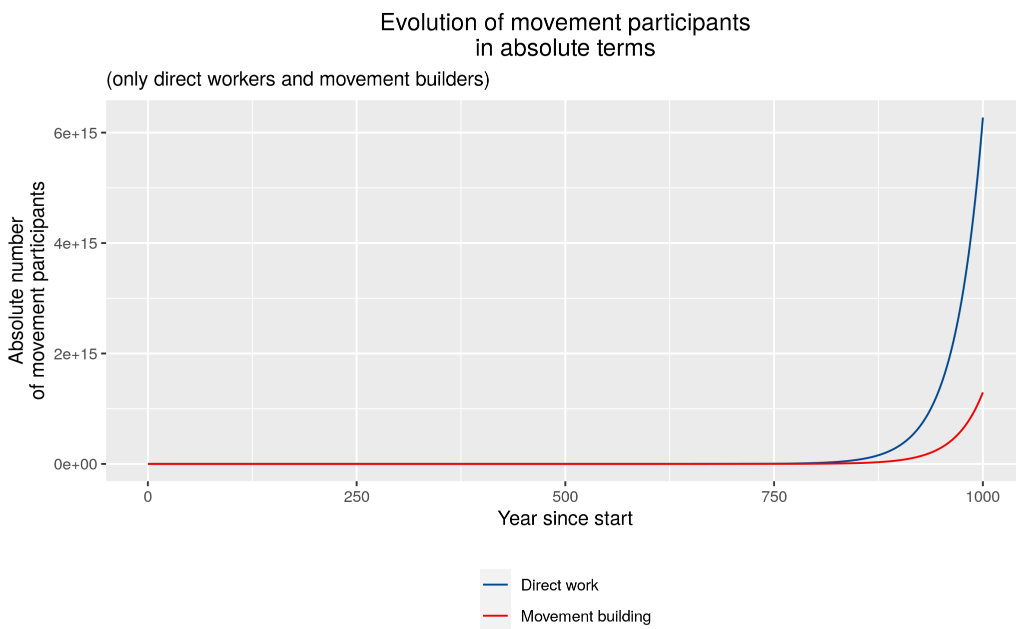

Evolution of state variables
(logarithmic scale)
1e+31
1e+24
1e+17
2e+10
Capital and labor
0
250
500
750
1000
Year since start
Capital
Labor

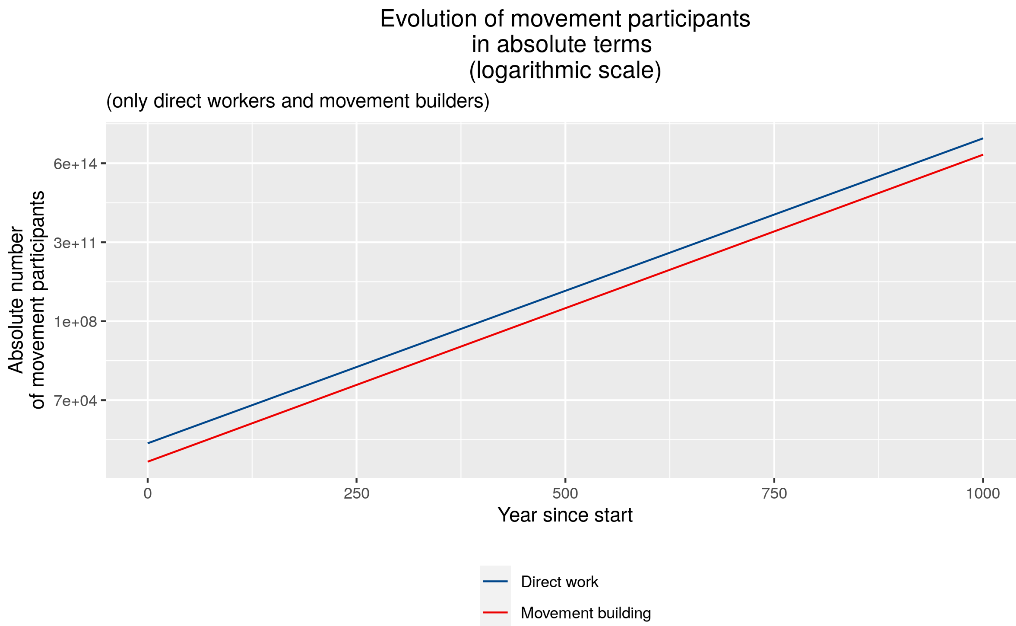

Evolution of movement size
Labor
2e+16
1e+16
5e+15
0e+00
0
250
500
750
1000
Year since start
Evolution of labor fractions
Fraction of total labor
1.0
0.8
0.5
0.2
0.0
0
250
500
750
1000
Year since start
Direct work
Movement building
Money-making


Evolution of labor fractions
(only direct work and movement building)
Fraction of total labor
0.4
0.3
0.2
0.1
0.0
0
250
500
750
1000
Year since start
Direct work
Movement building

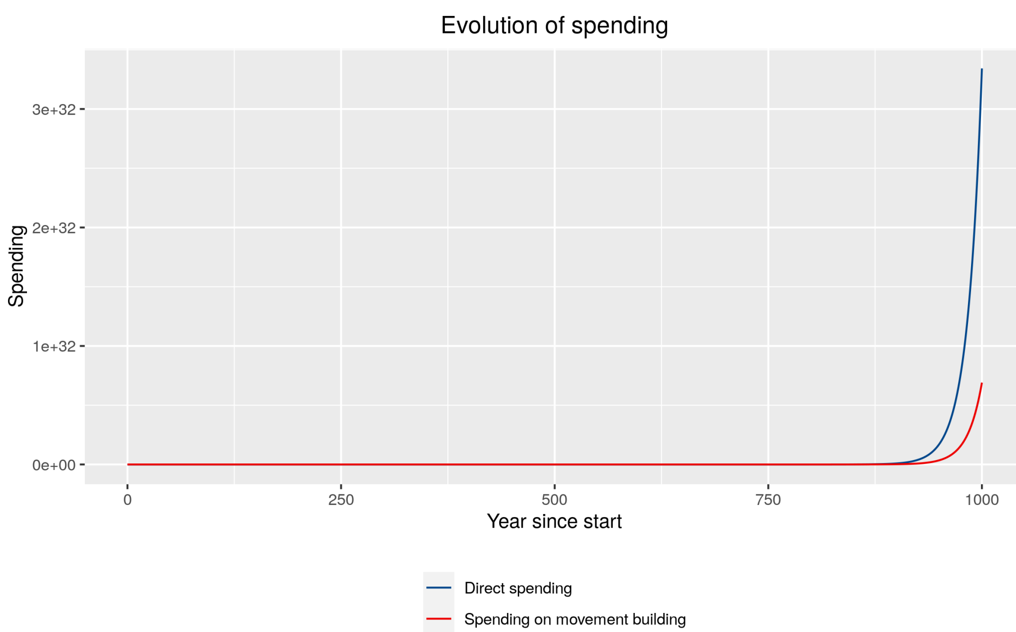

Evolution of movement building
as a fraction of total labor
Fraction of labor
0.08
0.06
0.04
0.02
0.00
0
250
500
750
1000
Year since start


Evolution of labor
in absolute terms
Absolute number
of movement participants
8e+15
6e+15
4e+15
2e+15
0e+00
0
250
500
750
1000
Year since start
Direct work
Movement building
Money-making

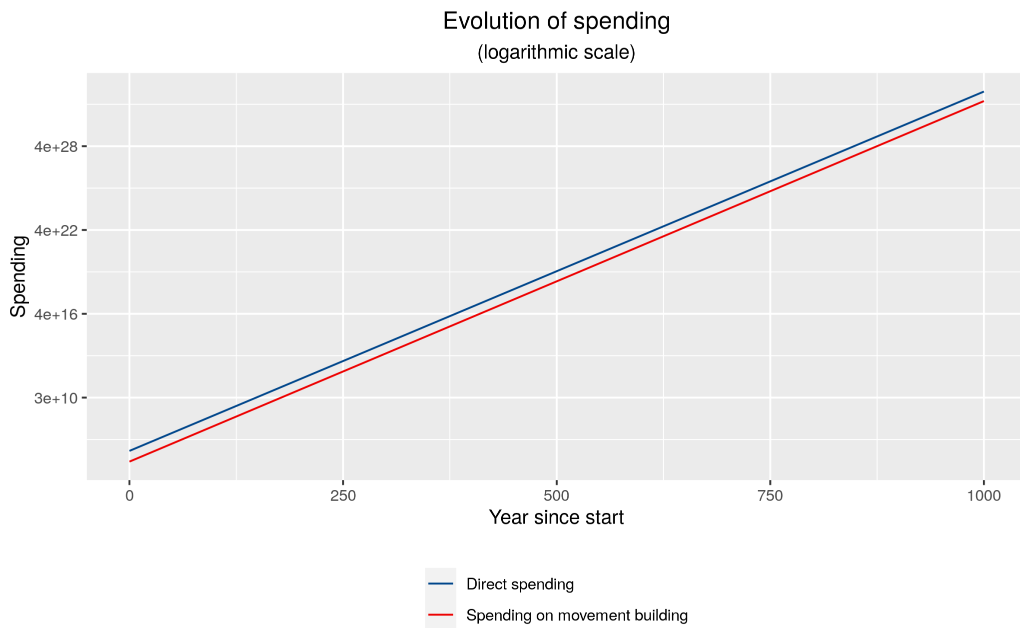

Evolution of movement participants
in absolute terms
(only direct workers and movement builders)
Absolute number
of movement participants
6e+15
4e+15
2e+15
0e+00
0
250
500
750
1000
Year since start
Direct work
Movement building



Evolution of movement participants
in absolute terms
(logarithmic scale)
(only direct workers and movement builders)
Absolute number
of movement participants
6e+14
3e+11
1e+08
7e+04
0
250
500
750
1000
Year since start
Direct work
Movement building

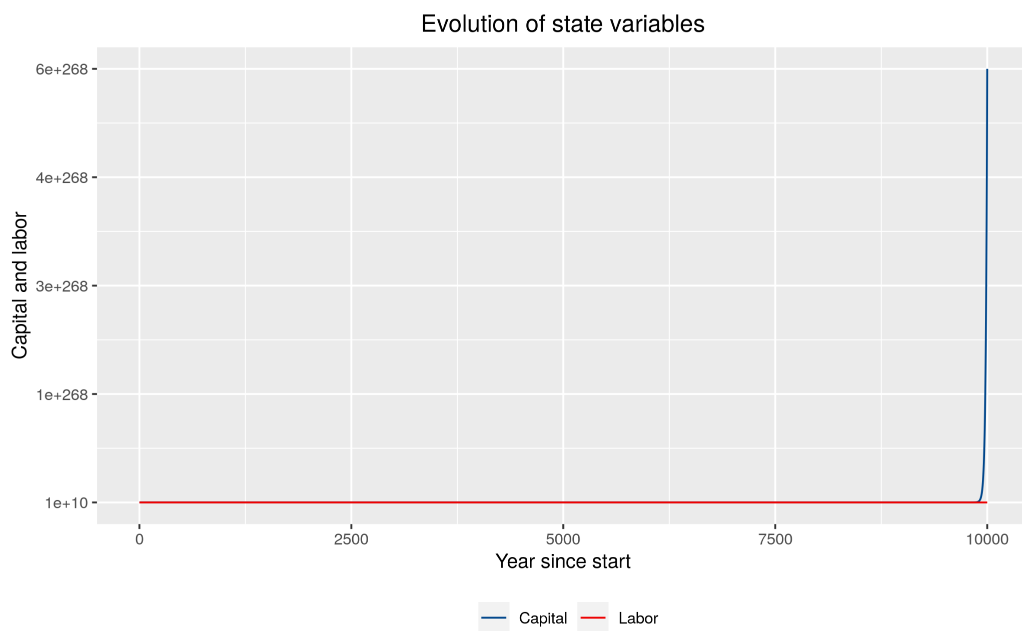


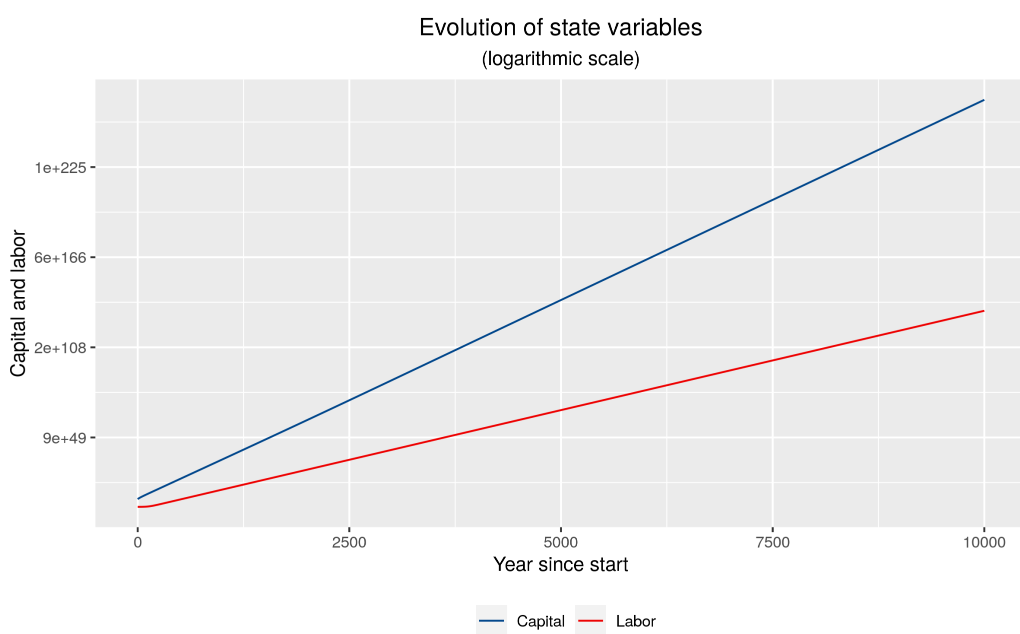
Evolution of spending
Spending
3e+32
2e+32
1e+32
0e+00
0
250
500
750
1000
Year since start
Direct spending
Spending on movement building

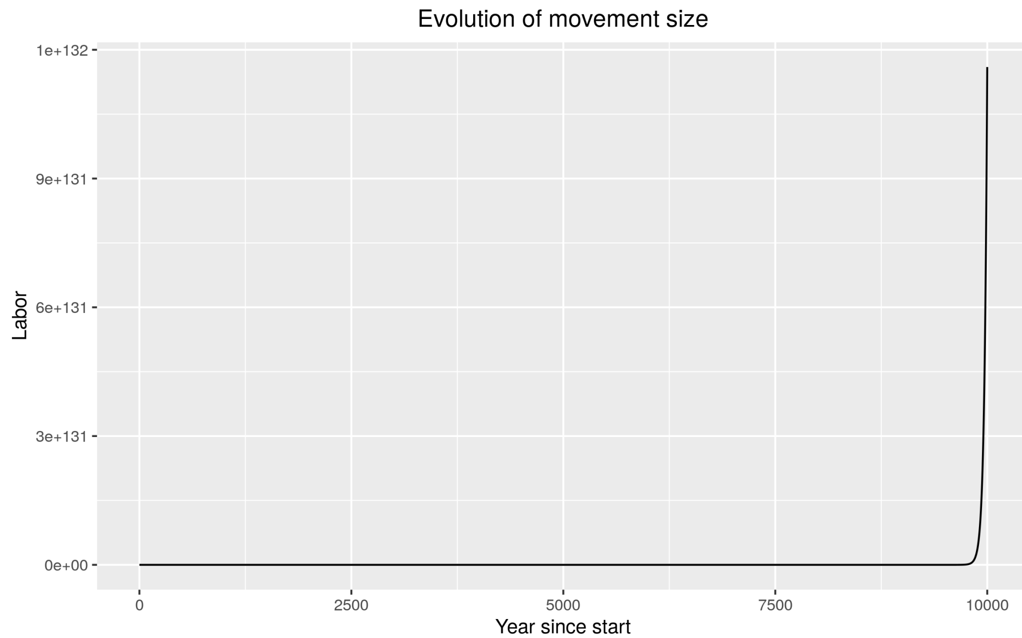


C.3 Graphical results: 10,000 years

Direct work as a fraction of movement size continues to decrease, perhaps exponentially, but doesn't yet go below movement building. However, we know from the balanced growth rates that it will do so. We can't display some of the graphs on a non-logarithmic scale due to large number limitations in R. [and I'm having some limitations in pushing forward the simulation much beyond 10k years]


Evolution of state variables
6e+268
4e+268
3e+268
1e+268
1e+10
Capital and labor
0
2500
5000
7500
10000
Year since start
Capital
Labor

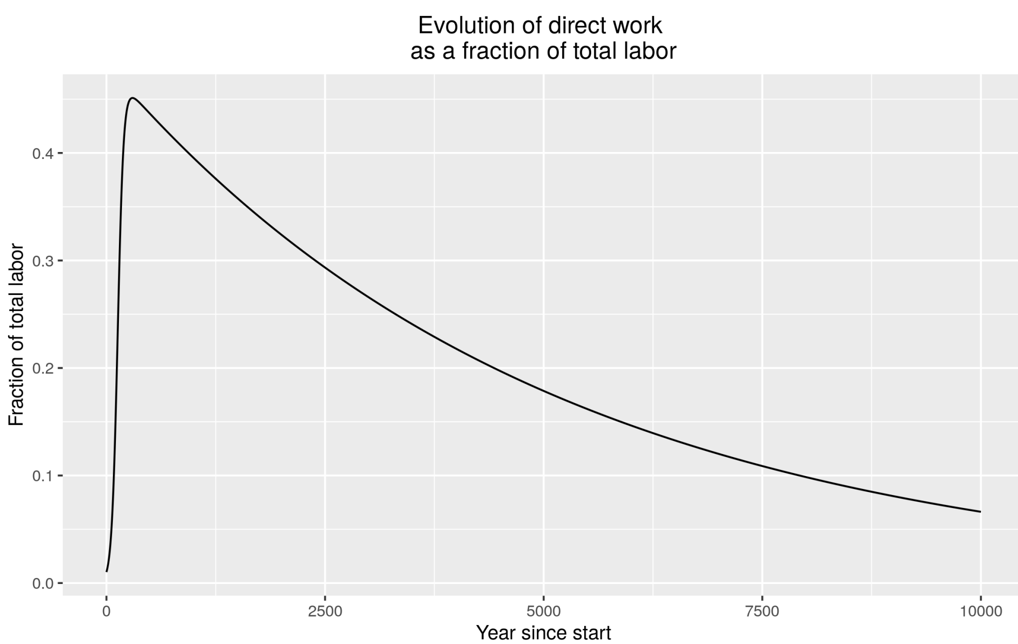

Evolution of state variables
(logarithmic scale)
1e+225
6e+166
2e+108
9e+49
Capital and labor
0
2500
5000
7500
10000
Year since start
Capital
Labor

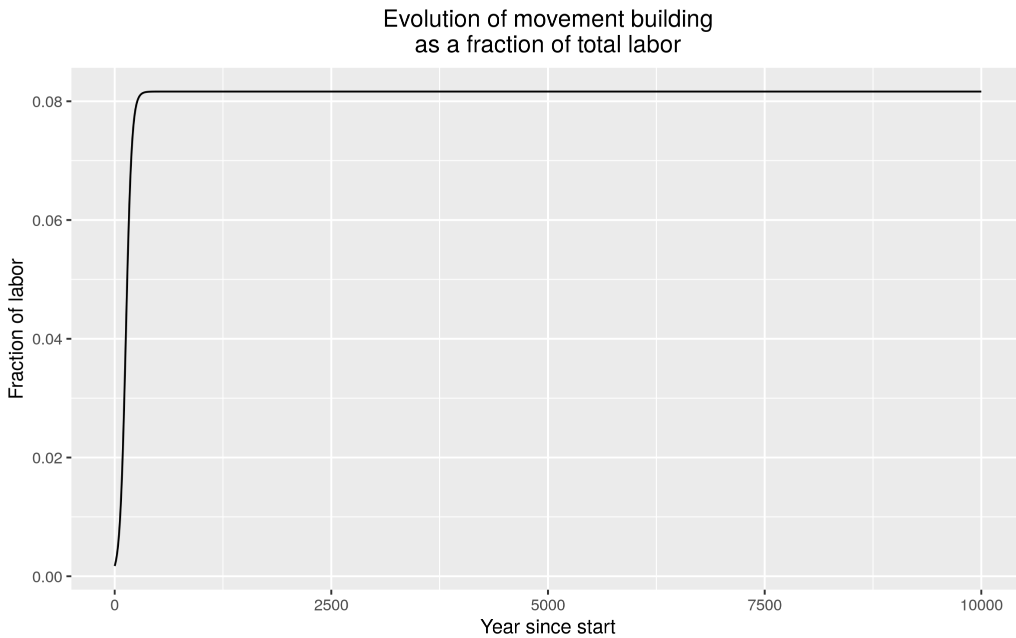


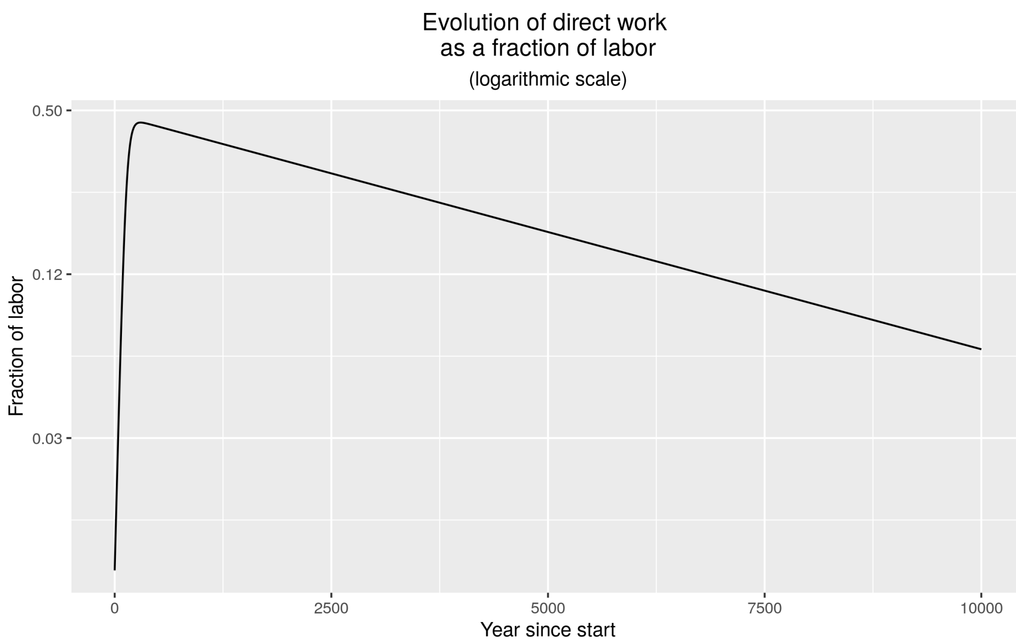
Evolution of movement size
1e+132
9e+131
6e+131
3e+131
0e+00
Labor
0
2500
5000
7500
10000
Year since start

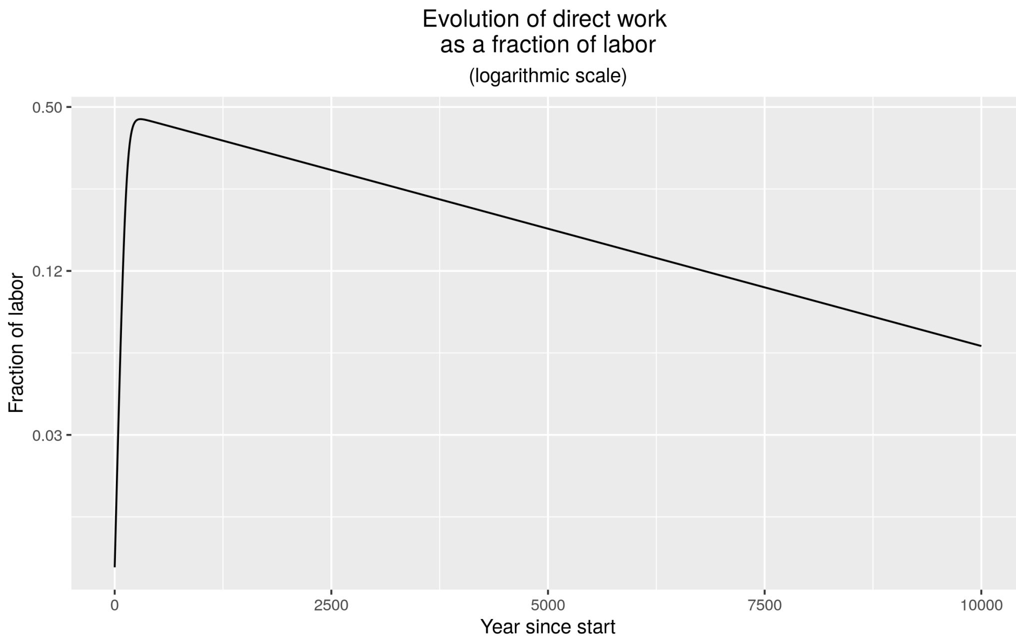
Evolution of labor fractions
1.0
0.8
0.5
0.2
0.0
Fraction of total labor
0
2500
5000
7500
10000
Year since start
Direct work
Movement building
Money-making


Evolution of labor fractions
(only direct work and movement building)
Fraction of total labor
0.0
0.1
0.2
0.3
0.4
0
2500
5000
7500
10000
Year since start
Direct work
Movement building

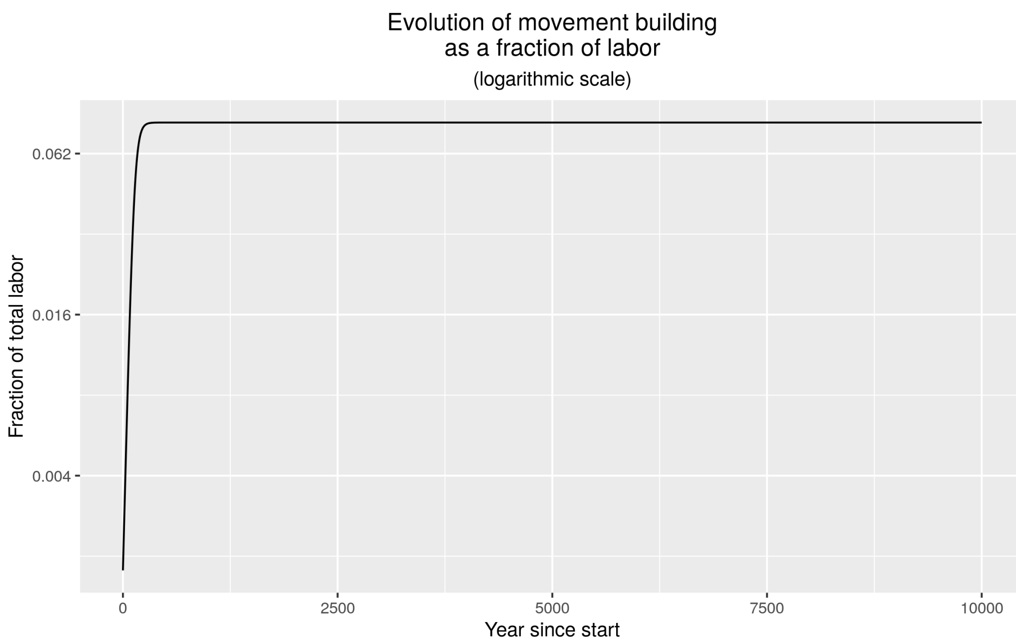

Evolution of direct work
as a fraction of total labor
Fraction of total labor
0.0
0.1
0.2
0.3
0.4
0
2500
5000
7500
10000
Year since start


Evolution of movement building
as a fraction of total labor
Fraction of labor
0.00
0.02
0.04
0.06
0.08
0
2500
5000
7500
10000
Year since start

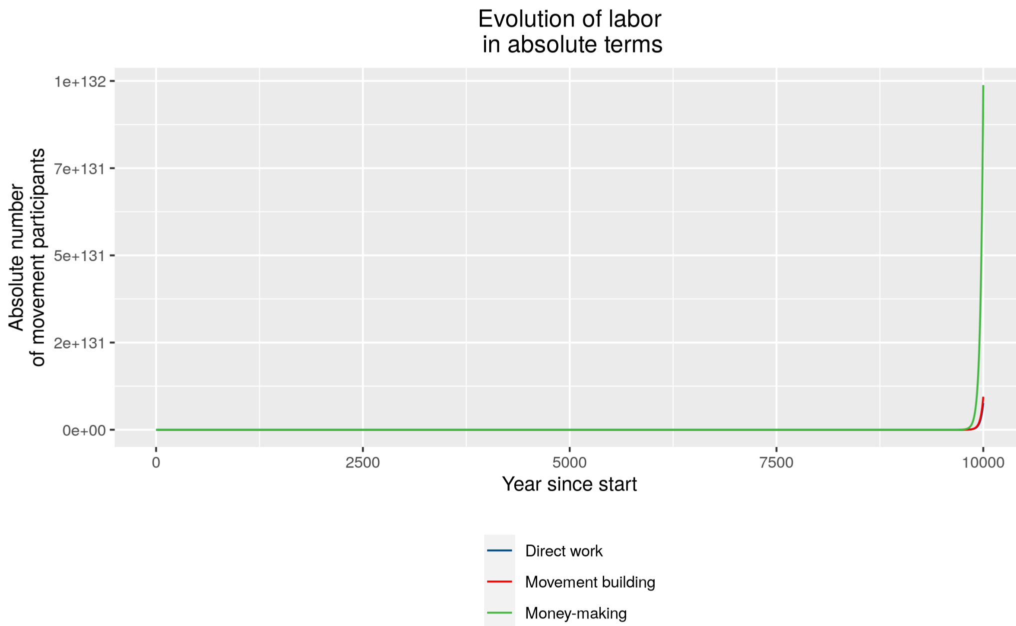

Evolution of direct work
as a fraction of labor
(logarithmic scale)
Fraction of labor
0.03
0.12
0.50
0
2500
5000
7500
10000
Year since start

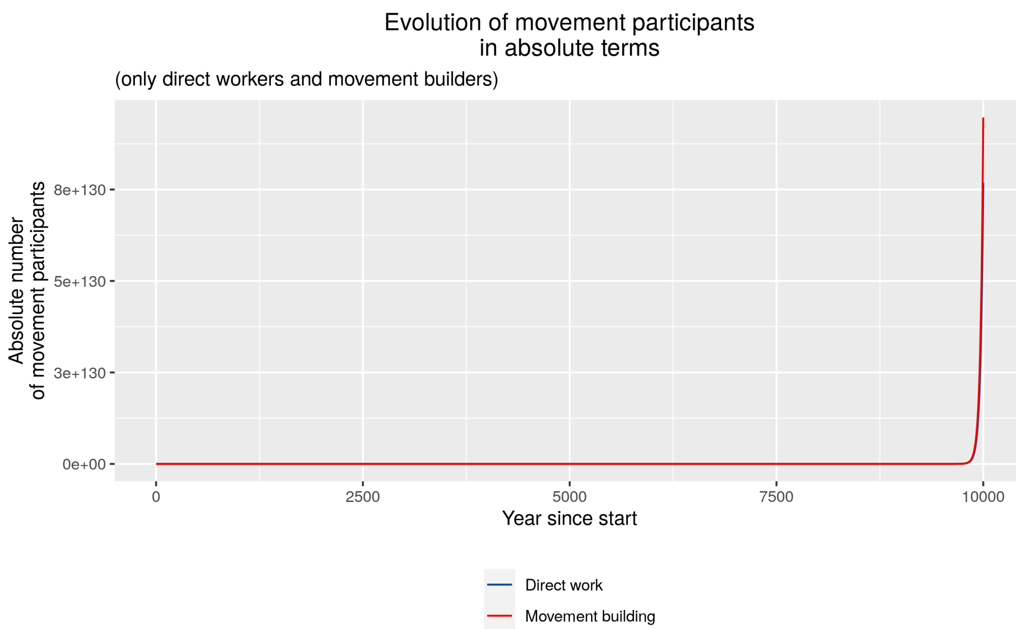


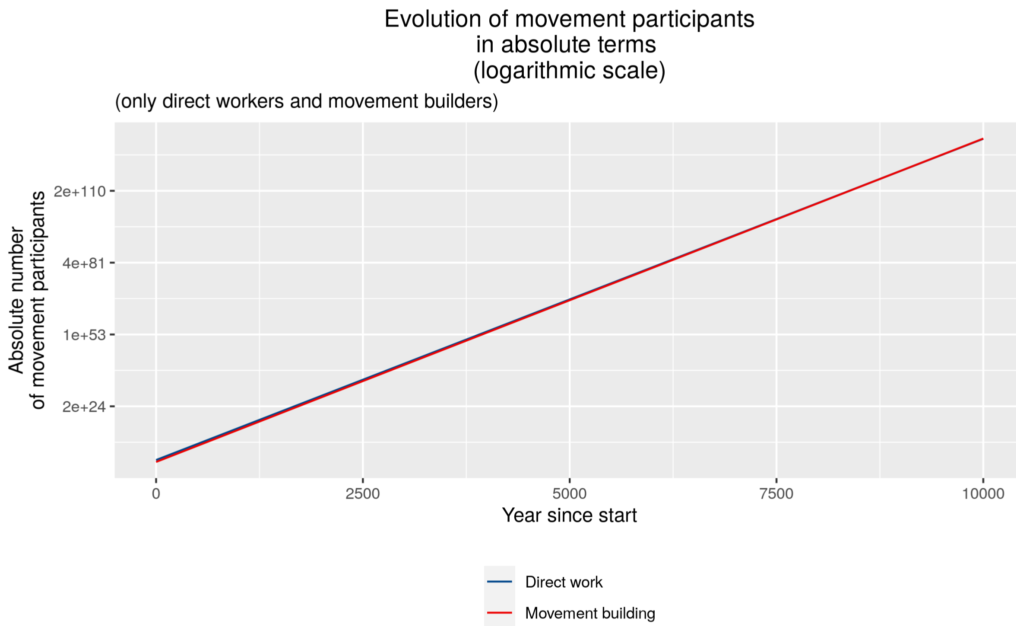
Evolution of direct work
as a fraction of labor
(logarithmic scale)
0.50
0.12
0.03
Fraction of labor
0
2500
5000
7500
10000
Year since start

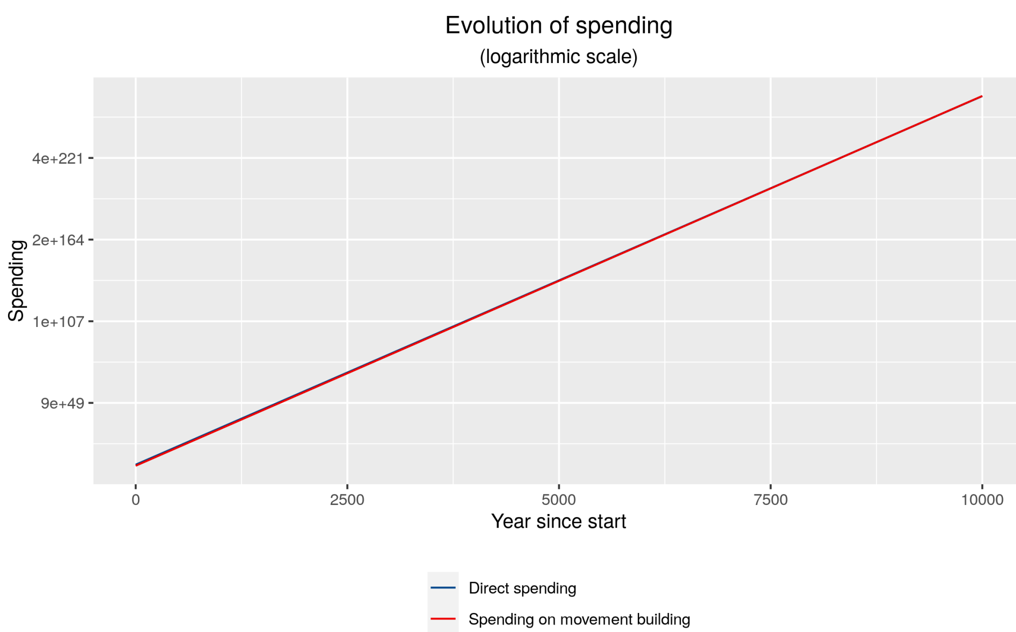


Evolution of movement building
as a fraction of labor
(logarithmic scale)
0.062
0.016
0.004
Fraction of total labor
0
2500
5000
7500
10000
Year since start


Evolution of labor
in absolute terms
Absolute number
of movement participants
1e+132
7e+131
5e+131
2e+131
0e+00
0
2500
5000
7500
10000
Year since start
Direct work
Movement building
Money-making


Evolution of movement participants
in absolute terms
(only direct workers and movement builders)
Absolute number
of movement participants
8e+130
5e+130
3e+130
0e+00
0
2500
5000
7500
10000
Year since start
Direct work
Movement building


Evolution of movement participants
in absolute terms
(logarithmic scale)
(only direct workers and movement builders)
Absolute number
of movement participants
2e+110
4e+81
1e+53
2e+24
0
2500
5000
7500
10000
Year since start
Direct work
Movement building


Evolution of spending
(logarithmic scale)
Spending
4e+221
2e+164
1e+107
9e+49
0
2500
5000
7500
10000
Year since start
Direct spending
Spending on movement building

D Why adding a $x_2^{\phi_2}$ to the law of motion for x_2 produces the same qualitative behavior in the limit

Equation (67) becomes:

$$\begin{aligned}\rho\mu_2 - \dot{\mu}_2 = \mu_2 \cdot (\rho - g_{\mu_2}) = (1-\eta)\cdot(1-\lambda_1)\cdot\frac{U}{x_2} \\ + \mu_1 \cdot w_2 \cdot exp\{\gamma_1 t\} \cdot (1-\sigma_1-\sigma_2) \\ + \mu_2 \cdot (1-\lambda_2)\cdot(\delta_2+\phi_2)\cdot\frac{F_2}{x_2}\end{aligned} \tag{126}$$

which simplifies to

$$\mu_2 \cdot (\rho - g_{\mu_2}) = \mu_1 \cdot w_2 \cdot exp\{\gamma_1 t\} \cdot \left(1 + \frac{\phi_2}{\delta_2}\cdot\sigma_2\right) \tag{127}$$

But the $\frac{\phi_2}{\delta_2}\cdot\sigma_2$ is at most a constant factor, so the balanced growth solution is the same.