Contents

Research and Movement Building for Utility Maximizers

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WORK IN PROGRESS

1 Introduction

Social movements such as "Effective Altruism" face the problem of optimal allocation of resources across time in order to maximize their desired impact. Much like states and other entities considered in the literature since (Ramsey, 1928) [1], they have the option to invest in order to give more later. They also have the option to conduct research in order to make their subsequent spending more effective. However, unlike states, where population dynamics are usually considered exogenous, such agents also have the option of recruiting like-minded associates through movement building. For example, Bill Gates can recruit other ultra-rich people through the *Giving Pledge*, aspiring effective altruists can likewise spread their ideas, etc.

Throughout this paper, we model the optimal allocation of funds between direct spending, investment, research, and movement building, as well as the optimal allocation of movement participants between direct workers, money earners, researchers, and movement builders. This research direction follows in the footsteps of (Trammel, 2020) [2], which considers a different facet of a related problem: the dynamics for a philanthropic funder who aims to provide public goods while having a lower discount than less patient partners.

The outline of our paper is as follows: In §2 we present the mathematical toolset and nomenclature used to find solutions for the optimal path problems, namely the Hamiltonian.

In §3 we consider a scenario where our utility function is isoelastic, and investment into research runs into diminishing returns. In this and subsequent sections, the full analytical solution is intractable, so we will study the balanced-growth path, a good approximation of the true solution in the limit. We'll also consider the ratios between the different variables of interest.

In §4 we come to consider a model which incorporates movement building, where a given movement starts out with a certain amount of money and a certain number of movement participants, and faces on the one hand a tradeoff between directly spending that money to generate utility, investing it, or spending that money on movement building, and on the other hand between directing the movement participants to directly work on generating utility, to work on generating money, or to work on movement building.

In §5 we combine the models in sections §4 and §3, and we find that previous results hold.

Lastly, we note with regret that our models are at present not exhaustive. To mention two omissions of particular relevance to the "Effective Altruism" movement, we don't consider global catastrophic or existential risks (such as runaway climate change, unaligned artificial intelligence, nuclear brinksmanship, extremely deadly global pandemics, etc.), which might lead us to consider more impatient allocations. On that note, we also don't here consider the interplay between players who have different rates of time discounting. Nonetheless, the stylized models we present are able to provide some insight by themselves, and moreover, might serve as building blocks for later models which take into account these and further considerations.

2 Setup

We're interested in the following general maximization problem

$$V(\vec{\alpha(t)}) = \max_{\vec{\alpha(t)}} \int_0^\infty e^{-\rho t} \cdot U(\vec{x(t)}, \vec{\alpha(t)})dt \tag{1}$$

Subject to

$$\dot{\vec{x}} = f(t, \vec{x(t)}, \vec{\alpha(t)}) \tag{2}$$

$$x_i \geq 0 \tag{3}$$

$$\vec{x}(0) = \vec{x_0}, \text{ given} \tag{4}$$

Where the variables stand for:

1. $\rho =$ Discount rate, perhaps value drift, risk of theft, etc.
2. $\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ ... \end{bmatrix} = \begin{bmatrix} \text{Capital at time } t \\ \text{Information/research at time } t \\ \text{Movement size at time } t \\ ... \end{bmatrix}$
3. $\vec{\alpha} = \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \alpha_3(t) \\ ... \end{bmatrix} = \begin{bmatrix} \text{Investment into altruistic projects at time } t \\ \text{Investment into research at time } t \\ \text{Investment into movement building at time } t \\ .. \end{bmatrix}$
4. $f =$ Function relating $\vec{x}$ and $\vec{\alpha}$
5. $U =$ Our utility function. A function from capital, information and current spending and other resources into altruistic impact

To find the optimal allocation path, we can define the current value Hamiltonian[1]:

$$H := U(\vec{x(t)}, \vec{\alpha(t)}) + \vec{\mu(t)}^{(T)} \cdot \dot{\vec{x}} \tag{5}$$

$$H = U(\vec{x(t)}, \vec{\alpha(t)}) + \mu(t)_1 \cdot \dot{x_1}(t) + \mu(t)_2 \cdot \dot{x_2}(t) \tag{6}$$

[1]Casual readers have repeatedly thought this referred to the Hamiltonian operator in quantum mechanics, or to one of the many other objects refered to as a "Hamiltonian". We note that we refer to the Hamiltonian in optimal control theory.

Our solution, that is, the optimal allocation path, will be given by the following constraints on the Hamiltonian:

$$\frac{\partial H}{\partial \vec{\alpha}} = 0 \tag{7}$$

$$-\frac{\partial H}{\partial \vec{x}} = \dot{\vec{\mu}} - \rho \vec{\mu} \tag{8}$$

$$-\frac{\partial H}{\partial \vec{\mu}} = \dot{\vec{x}} \tag{9}$$

Further, our solution must conform to the following transversality condition:

$$\lim_{t \to \infty} exp\{-\rho \cdot t\} \cdot (\vec{x} \cdot \vec{\mu}) = \vec{0} \tag{10}$$

or, in scalar form

$$\lim_{t \to \infty} exp\{-\rho \cdot t\} \cdot x_i \cdot \mu_i = 0 \tag{11}$$

We will drop the vector references and references to time from now on, while keeping them in mind.

The mathematically sophisticated reader without previous knowlesdge of optimal control theory who is looking for a short introduction to Hamiltonians, and to why they provide a solution to our maximization problem, is welcome to consult (Kurat 2013) [4]. For a minimal model to familiarize oneself with the concepts, consult either the basic model in (Trammel, 2020) [2], or a rather minimal model which nonetheless includes research in Appendix §A.

3 Research

3.1 Setup

The minimal model in Appendix §A falls prey to the ”nine women can't actually make a baby in a month” and the ”Rome wasn't built in a day” problems. That is, it assumes that spending \$X dollars on research all at once is as valuable as spending them throughout a longer period. In this section, we incorporate an elasticity to research spending, λ_2, to adress this problem.

We are maximizing:

$$V(\vec{\alpha(t)}) = \max_{\vec{\alpha(t)}} \int_0^\infty e^{-\rho t} \cdot U(\vec{x(t)}, \vec{\alpha(t)})dt \tag{12}$$

For utility and laws of motion:

$$U(x, \alpha) = \frac{(x_2 \cdot \alpha_1)^{1-\eta}}{1-\eta} \tag{13}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} r_1 x_1 - \alpha_1 - \alpha_2 \\ \beta_2 \cdot exp\{\gamma_2 t\} \cdot \alpha_2^{\lambda_2} \end{bmatrix} \tag{14}$$

under the constraints that

$$\lim_{t\to\infty} x_i \geq 0 \wedge x_2 \geq 0 \wedge \alpha_i \geq 0 \tag{15}$$

With our Hamiltonian standing at:

$$H = \frac{(x_2 \cdot \alpha_1)^{1-\eta}}{1-\eta} + \mu_1 \cdot (r_1 x_1 - \alpha_1 - \alpha_2) + \mu_2 \cdot (\beta_2 \cdot exp\{\gamma_2 t\} \cdot \alpha_2^{\lambda_2}) \tag{16}$$

and our transversality condition:

$$\lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot x_i \cdot \mu_i = 0 \tag{17}$$

Now, when $\lambda_2 < 1$, it will tend to be better to spend a research budget $\int exp\{-r_1\} \cdot \alpha_2(t)dt$ across many years rather than all at once.

3.2 Optimal path turns out to be intractable

By manipulating the constraints on the Hamiltonian, one ends up with the following system:

$$\begin{cases} \frac{x_2^{1-\eta}}{\alpha_1^{\eta}} = \mu_1 \\ \mu_1 = \mu_2 \cdot \beta_2 \cdot exp\{\gamma_2 t\}\lambda_2 \alpha_2^{(\lambda_2 - 1)} \\ \mu_1 = k_1 \cdot exp\{(\rho - r_1)t\} \\ -\frac{\alpha_1^{1-\eta}}{x_2^{\eta}} = \dot{\mu_2} - \rho\mu_2 \end{cases} \tag{18}$$

Now, this is somewhat intractable analytically. One solution strategy might be:

1. Subtitute α_1 for $f(x_2)$ using the first equation, so that $(\mu_2, \dot{\mu_2}) = g(x_2)$ in the last equation
2. Differentiate, so that $(\mu_2, \dot{\mu_2}, \ddot{\mu_2}) = g(\dot{x}_2) = h(\alpha_2)$
3. Substitute α_2 using the third equation.

However, this ends up with an analytically messy system, which appears intractable.

3.3 Balanced growth path derivation

Instead, we will find the *balanced growth path*, that is, the approximate solution to the system under the assumption that all variables grow at a constant exponential rate. This assumption is also a limit condition, so our solution, provided it satisfies the transversality condition, will become a better approximation in the limit.

That is, for every variable v in our system, we assume that $v = k_v \cdot exp\{g_v \cdot t\}$, with g_v constant. Under this assumption, ignoring the k_v constants, the system of equations at (18) transforms into:

$$\begin{cases} (1-\eta) \cdot g_{x_2} - \eta \cdot g_{\alpha_1} = g_{\mu_1} \\ g_{\mu_1} = g_{\mu_2} + \gamma_2 + (\lambda_2 - 1) \cdot g_{\alpha_2} \\ g_{\mu_1} = (\rho - r_1) \\ (1-\eta) \cdot g_{\alpha_1} - \eta \cdot g_{x_2} = g_{\mu_2} \end{cases} \tag{19}$$

And from equation (14) we can further obtain

$$g_{x_2} = \gamma_2 + \lambda_2 \cdot g_{\alpha_2} \tag{20}$$

But this is a solvable linear system, whose solution, under the constraints that $\lambda_2 \neq 1$ and $\lambda_2\eta - \lambda_2 + \eta \neq 0$, is:

$$\begin{cases} g_{\mu_1} = \rho - r_1, \\ g_{\mu_2} \cdot (\lambda_2\eta - \lambda_2 + \eta) = -[\gamma_2 \cdot (2\eta - 1) + (r_1 - \rho) \cdot (\lambda_2\eta + \eta - 1)] \\ g_{x_2} \cdot (\lambda_2\eta - \lambda_2 + \eta) = (\gamma_2\eta - \rho\lambda_2 + \lambda_2 r_1) \\ g_{\alpha_1} \cdot (\lambda_2\eta - \lambda_2 + \eta) = (r_1 + \gamma_2 \cdot (1 - \eta) - \rho) \\ g_{\alpha_2} \cdot (\lambda_2\eta - \lambda_2 + \eta) = (r_1 + \gamma_2 \cdot (1 - \eta) - \rho) \end{cases} \tag{21}$$

x_1 is given by it's law of motion (14). With θ_1 given by

$$\theta_1 = \frac{(r_1 + \gamma_2 \cdot (1 - \eta) - \rho)}{(\lambda_2\eta - \lambda_2 + \eta)} \tag{22}$$

then, with m_1, m_2 as of yet unknown constants,

$$x_1 = m_1 \exp\{r_1 t\} - m_2 \cdot exp\{\theta_1 t\} \tag{23}$$

Note that when $\theta_1 < r_1$, m_1 must be equal to 0, because we would otherwise accumulate a fortune which we would never spend, and our transversality condition would not be satisfied.

3.4 Variable ratios

Besides the growth factors g_v, we're also interested in the ratios between variables, and in particular between α_1 and α_2. Even though they may grow with the same exponent, one might still be smaller than the other.

From our initial equations (where, in the balanced growth path, $\dot{z} = g_z \cdot z$):

$$\begin{cases} \frac{x_2^{1-\eta}}{\alpha_1^{\eta}} = \mu_1 \\ \frac{\alpha_1^{1-\eta}}{x_2^{\eta}} = \mu_2 \cdot (\rho - g_{\mu_2}) \\ \mu_1 = \mu_2 \cdot \beta_2 \cdot exp\{\gamma_2 t\}\lambda_2\alpha_2^{(\lambda_2 - 1)} \\ g_{x_2} \cdot x_2 = \beta_2 \cdot exp\{\gamma_2\} \cdot \alpha_2^{\lambda_2} \end{cases} \tag{24}$$

We derive through repeated substitutions:

$$\frac{x_2^{1-\eta}}{\alpha_1^{\eta}} = \frac{\frac{\alpha_1^{1-\eta}}{x_2^{\eta}}}{(\rho - g_{\mu_2})} \cdot \lambda_2 \cdot \frac{g_{x_2} \cdot x_2}{\alpha_2} \tag{25}$$

which reduces to:

$$\frac{\alpha_1}{\alpha_2} = \frac{\rho - g_{\mu_2}}{\lambda_2 \cdot g_{x_2}} \tag{26}$$

3.5 Checking the transversality condition

The variables we need are

$$\mu_1 = k_1 \cdot exp\{(\rho - r_1) \cdot t\} \tag{27}$$

$$\mu_2 = \exp\left\{-\frac{\gamma_2 \cdot (2\eta - 1) + (r_1 - \rho) \cdot (\lambda_2\eta + \eta - 1)}{(\lambda_2\eta - \lambda_2 + \eta)} \cdot t\right\} \tag{28}$$

$$x_1 = m_1 \exp\{r_1 t\} - m_2 \cdot exp\left\{\frac{r_1 + \gamma_2 \cdot (1 - \eta) - \rho}{(\lambda_2\eta - \lambda_2 + \eta)} \cdot t\right\} \tag{29}$$

$$x_2 = k \cdot exp\left\{\frac{\gamma_2\eta - \rho\lambda_2 + \lambda_2 r_1}{\lambda_2\eta - \lambda_2 + \eta} \cdot t\right\} \tag{30}$$

The transversality condition is

$$\lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot x_i \cdot \mu_i = 0 \tag{31}$$

For $i = 1$, $i = 2$, this holds (after some algebra which outputs the same inequality in both cases) when:

$$r_1 \cdot \left(1 - \frac{1}{\lambda_2 \cdot \eta - \lambda_2 + \eta}\right) > \frac{\gamma_2 \cdot (1 - \eta) - \rho}{\lambda_2 \cdot \eta - \lambda_2 + \eta} \tag{32}$$

For $i = 1$, we additionally require that $m_1 = 0$.

3.6 Results and interpretation

$$\theta_1 = \frac{(r_1 + \gamma_2 \cdot (1-\eta) - \rho)}{(\lambda_2\eta - \lambda_2 + \eta)} \tag{33}$$

θ_1 corresponds to the growth rate for both direct spending (α_1) and spending on research (α_2). As an example scenario, consider:

$$\begin{cases} \lambda_2 = \text{elasticity of research} = 0.5 \\ \eta = \text{elasticty of spending} = 1.1 \\ \rho = \text{hazard rate} = 0.005 = 0.5\% \\ r_1 = \text{returns above inflation} = 0.06 = 6\% \\ \gamma_2 = \text{independent, unpaid-for research} = 0.01 = 1\% \end{cases} \tag{34}$$

Note that this satisfies the inequality in (32):

$$0.06 \cdot \left(1 - \frac{1}{0.5 \cdot 1.1 - 0.5 + 1.1}\right) > \frac{0.01 \cdot (1 - 1.1) - 0.005}{0.5 \cdot 1.1 - 0.5 + 1.1} \tag{35}$$

which holds, and leaves θ_1 to be

$$\theta_1 = \frac{(0.06 + 0.01 \cdot (1 - 1.1) - 0.005)}{(0.5 \cdot 1.1 - 0.5 + 1.1)} = 27/575 \tag{36}$$

$$\theta_1 \approx 0.04695652... \approx 0.047 \tag{37}$$

Note that this is not 4.7% of the total budget, it's saying that direct spending and spending in research grow with a growth factor of 4.7%.

If instead $\eta = 0.9$, then the transversality condition doesn't hold.

Once r_1 and θ_1 are known, we can determine what proportion of our funds we spend year on year. First, an example: if $\theta_1 = 5\% = 0.05$, then x_1 will grow at a rate of θ_1, while subject to r_1 returns, then:

$$(1 + 0.06) \cdot x_1 + q \cdot x = (1 + 0.05) \cdot x_1 \implies q = 0.01 = 1\% \tag{38}$$

So in this example we would spend 1% of our budget per year. The general reasoning is that if our spending is growing at a rate of θ_1 per year, then our fortune must also be growing at that rate: if it was growing more slowly, we'd at some point run out of money, and if it was growing at a faster rate, we'd accumulate money which we'd never spend. And this uniquely determines the fraction of our fortune q to spend each time period.

We can also look at the ratio between $\alpha_1(t)$ and $\alpha_2(t)$.

$$\frac{\alpha_1}{\alpha_2} = \frac{\rho - g_{\mu_2}}{\lambda_2 \cdot g_{x_2}} \tag{39}$$

For the same variable values as in (34), that factor is approximately equal to 2.77922. This corresponds to a substantial fraction of spending being directed towards research, but this is sensitive to our choice of λ_2, which might experimentally be determined to be much lower than 0.5. This result will change once we consider research in conjunction with movement building.

4 Movement building

4.1 Setup

When adding movement building to our model, we'll add some new variables for consideration:

1. x_3, total movement size, and α_3, the money spent on movement building on a given instant. These are analogous to x_1, x_2, α_1 and α_2.
2. $\sigma_1, \sigma_2, \sigma_3$: the fraction of the movement which works respectively on direct work, research and movement building.
3. $w_3 \cdot exp\{\gamma_1 t\}$: wages rising with economic growth, and $\beta_3 \cdot exp\{\gamma_3 t\}$: the changing difficulty of recruiting movement participants. γ_3 might be hypothesized to be negative, given that economic growth provides better outside options, but empirically seems to be positive. For simplicity, we will consider these rates —γ_1 and γ_3— to be exogenous.
4. δ_3: elasticity of movement building with movement size.

For the remaining of this section, we'll forget about research $(x_2, \alpha_2, \sigma_2)$, in order to first get some preliminary results and intuitions about movement building. Now, we are maximizing:

$$V(\alpha(\vec{t})) = \max_{\alpha(\vec{t})} \int_0^\infty e^{-\rho t} \cdot U(x(\vec{t}), \alpha(\vec{t}))dt \tag{40}$$

For utility and laws of motion:

$$U(x, \alpha) = \frac{(\alpha_1^{\lambda_1}(\sigma_1 x_3)^{1-\lambda_1})^{1-\eta}}{1-\eta} \tag{41}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} r_1 x_1 - \alpha_1 - \alpha_3 + x_3 \cdot w_3 \cdot exp\{\gamma_1 t\} \cdot (1 - \sigma_1 - \sigma_3) \\ \beta_3 \cdot exp\{\gamma_3 t\} \cdot (\alpha_3^{\lambda_3} \cdot (\sigma_3 x_3)^{1-\lambda_3})^{\delta_3} \end{bmatrix} \tag{42}$$

under the constraints that

$$\lim_{t\to\infty} x_i \geq 0 \wedge x_3 \geq 0 \wedge \alpha_i \geq 0 \wedge \sigma_1 + \sigma_3 \leq 1 \tag{43}$$

With our Hamiltonian standing at:

$$H = U + \mu_1 \cdot \dot{x}_1 + \mu_3 \cdot \dot{x}_3 \tag{44}$$

$$\begin{aligned} H = &\frac{(\alpha_1^{\lambda_1}(\sigma_1 x_3)^{1-\lambda_1})^{1-\eta}}{1-\eta} \\ &+ \mu_1 \cdot (r_1 x_1 - \alpha_1 - \alpha_3 + x_3 \cdot w_3 \cdot exp\{\gamma_1 t\} \cdot (1 - \sigma_1 - \sigma_3)) \\ &+ \mu_3 \cdot (\beta_3 \cdot exp\{\gamma_3 t\} \cdot (\alpha_3^{\lambda_3} \cdot (\sigma_3 x_3)^{1-\lambda_3})^{\delta_3}) \end{aligned} \tag{45}$$

and our transversality condition same as it ever was:

$$\lim_{t \to \infty} exp\{-\rho \cdot t\} \cdot x_i \cdot \mu_i = 0 \tag{46}$$

For convenience, $F_3 := \beta_3 \cdot (\alpha_3^{\lambda_3} \cdot (\sigma_3 x_3)^{1-\lambda_3})^{\delta_3}$. Note that $F_3 = \dot{x}_3$

4.2 Hamiltonian equations

4.2.1 $\frac{\partial H}{\partial \alpha_1} = 0$

$$(1-\eta) \cdot \lambda_1 \cdot \frac{U}{\alpha_1} - \mu_1 = 0 \tag{47}$$

$$\mu_1 = (1-\eta)\lambda_1 \cdot \frac{U}{\alpha_1} \tag{48}$$

4.2.2 $\frac{\partial H}{\partial \alpha_3} = 0$

$$\mu_3 \cdot \delta_3 \lambda_3 \cdot \frac{F_3}{\alpha_3} - \mu_1 = 0 \tag{49}$$

$$\mu_1 = \mu_3 \cdot \delta_3 \cdot \lambda_3 \cdot \frac{F_3}{\alpha_3} \tag{50}$$

4.2.3 $\frac{\partial H}{\partial \sigma_1} = 0$

$$(1-\eta)(1-\lambda_1) \cdot \frac{U}{\sigma_1} - \mu_1 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\} = 0 \tag{51}$$

$$\mu_1 = \frac{(1-\eta)(1-\lambda_1)}{w_3} \cdot \frac{U}{\sigma_1 \cdot x_3 \cdot exp\{\gamma_1 t\}} \tag{52}$$

4.2.4 $\frac{\partial H}{\partial \sigma_3} = 0$

$$-\mu_1 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\} + \mu_3 \cdot \delta_3(1-\lambda_3) \cdot \frac{F_3}{\sigma_3} = 0 \tag{53}$$

$$\mu_1 = \mu_3 \cdot \frac{\delta_3 \cdot (1-\lambda_3)}{w_3} \cdot \frac{F_3}{\sigma_3 \cdot x_3 \cdot exp\{\gamma_1 t\}} \tag{54}$$

4.2.5 $\frac{\partial H}{\partial x_1} = \rho\mu_1 - \dot{\mu}_1$

$$\mu_1 \cdot r_1 = \rho\mu_1 - \dot{\mu}_1 \tag{55}$$

$$\mu_1 = k_1 \cdot exp\{(\rho - r_1)t\} \tag{56}$$

4.2.6 $\frac{\partial H}{\partial x_3} = \rho\mu_3 - \dot{\mu}_3$

$$\begin{aligned} \rho\mu_3 - \dot{\mu}_3 = \mu_3 \cdot (\rho - g_{\mu_3}) = (1-\eta) \cdot (1-\lambda_1) \cdot \frac{U}{x_3} \\ + \mu_1 \cdot w_3 \cdot exp\{\gamma_1 t\} \cdot (1 - \sigma_1 - \sigma_3) \\ + \mu_3 \cdot (1-\lambda_3) \cdot \delta_3 \cdot \frac{F_3}{x_3} \end{aligned} \tag{57}$$

Through several manipulations of (57), in particular by substituting $(1-\eta) \cdot (1-\lambda_1) \cdot U$ from (52) and $(1-\lambda_1) \cdot \delta_3 \cdot F_3 \cdot \mu_3$ from (54), we arrive at:

$$\mu_3 \cdot (\rho - g_{\mu_3}) = \mu_1 \cdot w_3 \cdot exp\{\gamma_1 t\} \tag{58}$$

This produces the growth equation

$$g_{\mu_1} = g_{\mu_3} + g_{x_3} - \gamma_1 \tag{59}$$

4.2.7 Summary

$$\mu_1 = (1-\eta)\lambda_1 \cdot \frac{U}{\alpha_1} \tag{60}$$

$$\mu_1 = \mu_3 \cdot \delta_3 \cdot \lambda_3 \cdot \frac{F_3}{\alpha_3} \tag{61}$$

$$\mu_1 = \frac{(1-\eta)(1-\lambda_1)}{w_3} \cdot \frac{U}{\sigma_1 \cdot exp\{\gamma_1 t\}} \tag{62}$$

$$\mu_1 = \mu_3 \cdot \frac{\delta_3 \cdot (1-\lambda_3)}{w_3} \cdot \frac{F_3}{\sigma_3 \cdot exp\{\gamma_1 t\}} \tag{63}$$

$$\mu_1 = k_1 \cdot exp\{(\rho - r_1)t\} \tag{64}$$

$$\mu_3 \cdot (\rho - g_{\mu_3}) = \mu_1 \cdot w_3 \cdot exp\{\gamma_1 t\} \tag{65}$$

4.3 Variable ratios

By dividing (60) by (62) and (61) by (63), we conclude that:

$$\frac{\lambda_1}{\alpha_1} = \frac{1-\lambda_1}{\sigma_1 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\}} \tag{66}$$

$$\frac{\lambda_3}{\alpha_3} = \frac{1-\lambda_3}{\sigma_3 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\}} \tag{67}$$

and hence

$$\frac{(1-\lambda_1)}{\lambda_1} \cdot \frac{\alpha_1}{\sigma_1} = \frac{(1-\lambda_3)}{\lambda_3} \cdot \frac{\alpha_3}{\sigma_3} \tag{68}$$

We can also get some mileage out of considering the law of motion for x_3, (42), in conjunction with (67) and the fact that in the balanced growth path, when $g_{x_3} > 0$, $\dot{x_3} = g_{x_3} \cdot x_3$. If we do so, we can isolate σ_3

$$\sigma_3 = \frac{g_{x_3}}{\beta_3} \cdot \left(\frac{1-\lambda_3}{\lambda_3 \cdot w_3}\right)^{\lambda_3 \delta_3} \cdot \frac{1}{\exp\{\gamma_3 + \gamma_1 \cdot \lambda_3 \cdot \delta_3\} \cdot x_3^{\delta_3}} \tag{69}$$

The last fraction converges to a constant on the balanced growth path.

Now, (68) is important enough that we will rederive it from the Euler equations, that is, just from the constraint that on the optimal path, the marginal value of moving people and funds around should be equal to 0. In particular,

$$\frac{\partial U}{\partial \$} = \frac{\partial U}{\partial people} \cdot \frac{\partial people}{\partial \$}_{\text{bought out of money-making}} \tag{70}$$

$$\frac{\partial people}{\partial \$}_{\text{through movement building}} = \frac{\partial people}{\partial people} \cdot \frac{\partial people}{\partial \$}_{\text{bought out of money-making}} \tag{71}$$

Equation (70) reads as "the *marginal* money-maker should produce as much value by making money and directly donating their earnings as by working directly." Equation (70) reads as "the *marginal* money-maker should create as many movement participants by making money and donating their earnings to movement building as by working on movement building themselves." Otherwise, we could move direct workers or movement builders towards money-making, or vice-versa.

From (42) and (44), our model definition, these two equations develop into:

$$\lambda_1 \cdot (1-\eta) \cdot \frac{U}{\alpha_1} = \left((1-\lambda_1) \cdot (1-\eta) \cdot \frac{U}{\sigma_1 \cdot x_3}\right) \cdot \left(\frac{1}{w_3 \cdot exp\{\gamma_1 \cdot t\}}\right) \tag{72}$$

$$\lambda_3 \cdot \delta_3 \cdot \frac{F_3}{\alpha_3} = \left((1-\lambda_3) \cdot \delta_3 \cdot \frac{F_3}{\sigma_3 \cdot x_3}\right) \cdot \left(\frac{1}{w_3 \cdot exp\{\gamma_1 \cdot t\}}\right) \tag{73}$$

Which simplify into

$$\frac{\lambda_1}{\alpha_1} = \frac{1-\lambda_1}{\sigma_1 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\}} \tag{74}$$

$$\frac{\lambda_3}{\alpha_3} = \frac{1-\lambda_3}{\sigma_3 \cdot x_3 \cdot w_3 \cdot exp\{\gamma_1 t\}} \tag{75}$$

i.e., (66) and (67), from which (68) follows:

$$\frac{(1-\lambda_1)}{\lambda_1} \cdot \frac{\alpha_1}{\sigma_1} = \frac{(1-\lambda_3)}{\lambda_3} \cdot \frac{\alpha_3}{\sigma_3} \tag{76}$$

We can understand this equation as a convenient necessary but not sufficient heuristic, such that a spending schedule which doesn't satisfy it suffers from the affliction that, insofar as our model is accurate enough, one would be able to obtain a better outcome by redistributing people and funds around.

4.4 Balanced growth equations

4.4.1 Balanced growth equations I

$$g_{\mu_1} = g_U - g_{\alpha_1} \tag{77}$$

$$g_{\mu_1} = g_{\mu_3} + g_{F_3} - g_{\alpha_3} \tag{78}$$

$$g_{\mu_1} = g_U - g_{\sigma_1} - g_{x_3} - \gamma_1 \tag{79}$$

$$g_{\mu_1} = g_{\mu_3} + g_{F_3} - g_{\sigma_3} - g_{x_3} - \gamma_1 \tag{80}$$

$$g_{\mu_1} = (\rho - r_1) \tag{81}$$

$$g_{\mu_1} = g_{\mu_3} - \gamma_1 \tag{82}$$

$$g_{x_3} = g_{F_3} = \gamma_3 + \delta_3 \cdot \Big(\lambda_3 \cdot g_{\alpha_3} + (1 - \lambda_3) \cdot (g_{\sigma_3} + g_{x_3}) \Big) \tag{83}$$

4.4.2 Balanced growth equations II

Some simple simplifications. (89) is derived from (80) + (82) + ($g_{x_3} = g_{F_3}$).

$$g_{\alpha_1} = g_{\sigma_1} + g_{x_3} + \gamma_1 \tag{84}$$

$$g_{\mu_1} = g_U - g_{\alpha_1} \tag{85}$$

$$g_{\alpha_3} = g_{\sigma_3} + g_{x_3} + \gamma_1 \tag{86}$$

$$g_{\mu_1} = g_{\mu_3} + g_{F_3} - g_{\alpha_3} \tag{87}$$

$$g_{\mu_1} = \rho - r_1 \tag{88}$$

$$g_{\sigma_3} = 0 \tag{89}$$

$$g_{x_3} = g_{F_3} = \gamma_3 + \delta_3 \cdot \Big(\lambda_3 \cdot g_{\alpha_3} + (1 - \lambda_3) \cdot (g_{\sigma_3} + g_{x_3})\Big) \tag{90}$$

4.5 Balanced growth path derivation

From this we can simply derive g_{x_3}, by substituting (86) and (89) in (90)

$$g_{x_3} = \frac{\gamma_3 + \delta_3 \lambda_3 \gamma_1}{1 - \delta_3} \tag{91}$$

And from that g_{α_3}, by substituting (91) back in (86)

$$g_{\alpha_3} = g_{x_3} + \gamma_1 = \frac{\gamma_3 + \delta_3 \lambda_3 \gamma_1}{1 - \delta_3} + \gamma_1 \tag{92}$$

Similarly, from (84), (85) and (91), we can derive g_{α_1} and g_{σ_1}:

$$g_{\alpha_1} = \frac{r_1 - \rho}{\eta} - \frac{(1 - \eta)(1 - \lambda_1)}{\eta} \cdot \gamma_1 \tag{93}$$

$$g_{\sigma_1} = \frac{r - \rho}{\eta} - \left(\frac{(1 - \eta)(1 - \lambda_1)}{\eta} + 1\right) \cdot \gamma_1 - g_{x_3} \tag{94}$$

Note that this solution is only valid where $g_{\sigma_1} \leq 0$.

Note also that $g_{\alpha_1} \leq g_{\alpha_3}$. Proof: $g_{\alpha_1} = g_{\sigma_1} + g_{x_3} + \gamma_1$, and $g_{\alpha_3} = \cancel{g_{\sigma_3}} + g_{x_3} + \gamma_1$. Hence $g_{\alpha_1} = g_{\sigma_1} + g_{\alpha_3} \wedge g_{\sigma_1} \leq 0 \implies g_{\alpha_1} \leq g_{\alpha_3}$.

We can also derive x_1.

$$\dot{x}_1 = r_1 x_1 - \alpha_1 - \alpha_3 + x_3 \cdot w_3 \cdot exp\{\gamma_1 t\} \cdot (1 - \sigma_1 - \sigma_3) \tag{95}$$

$$x_1 = a \cdot exp\{r_1 \cdot t\} + b \cdot exp\{g_{\alpha_1} \cdot t\} + c \cdot exp\{g_{\alpha_3} \cdot t\} \tag{96}$$

4.6 Checking the transversality condition

The variables we need follow. We get μ_3 from (80) + ($g_{F_3} = g_{x_3}$) + ($g_{\sigma_3} = 0$)

$$\mu_1 = k_1 \cdot exp\{(\rho - r_1) \cdot t\} \tag{97}$$

$$\mu_3 = exp\left\{\Big((\rho - r_1) + \gamma_1\Big) \cdot t\right\} \tag{98}$$

$$x_1 = a \cdot exp\{r_1 \cdot t\} + b \cdot exp\left\{\left(\frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta} \cdot \gamma_1\right) \cdot t\right\} + c \cdot exp\{g_{\alpha_3} \cdot t\} \tag{99}$$

$$x_3 = exp\left\{\frac{\gamma_3 + \delta_3 \lambda_3 \gamma_1}{1 - \delta_3} \cdot t\right\} \tag{100}$$

The transversality condition is

$$\lim_{t \to \infty} exp\{-\rho \cdot t\} \cdot x_i \cdot \mu_i = 0 \tag{101}$$

For $i = 1$, this requires $a = 0$. In the case where $c = 0$,

$$-\rho + (\rho - r_1) + \left(\frac{r_1 - \rho}{\eta} - \frac{(1-\eta) \cdot (1-\lambda_1)}{\eta} \cdot \gamma_1\right) < 0 \tag{102}$$

which simplifies to

$$-\frac{\rho}{\eta} - \frac{(1-\eta) \cdot (1-\lambda_1)}{\eta} \cdot \gamma_1 < r_1 \cdot \left(1 - \frac{1}{\eta}\right) \tag{103}$$

or, alternatively, to

$$g_{\alpha_1} = g_{x_1} < r_1 \tag{104}$$

For $\rho \approx 0.005, \gamma_1 \approx 0.05, \gamma_3 \approx 0.01, r_1 \approx 0.06, \lambda_1 \approx 0.5$, this implies $\eta \gtrsim 0.86$.

What does $c = 0$ imply? From (95) and (96), it implies that

$$\alpha_3 = x_3 \cdot w_3 \cdot exp\{\gamma_1 t\} \cdot (1 - \sigma_1 - \sigma_3) \tag{105}$$

that is, that spending on movement building is equal movement participant contributions. From this and (67), we can further derive the long-run value of σ_3, for the cases where $g_{\sigma_1} < 0$, and this value is $1 - \lambda_3$:

$$\begin{cases} \alpha_3 = x_3 \cdot w_3 \cdot exp\{\gamma_1 t\} \cdot (1 - \cancel{\sigma_1} - \sigma_3) \\ (67) : \dfrac{\alpha_3}{\lambda_3} = \dfrac{\sigma_3 \cdot x_3 \cdot w_3 exp\{\gamma_1\}}{1 - \lambda_3} \end{cases} \implies \sigma_3 = 1 - \lambda_3 \tag{106}$$

With $a = 0, c = 0$, $g_{x_1} = g_{\alpha_1}$, and so one would spend $(r_1 - g_{\alpha_1})\%$ of x_1 on α_1, year on year, per reasoning similar to (38). We would be on a balanced growth path for all variables, and this balanced growth path satisfies the transversality conditions, thus the balanced growth path would be optimal, per (Romer, 1986) [5].

For $i = 3$, the transversality condition is satisfied when:

$$-\rho + (\rho - r_1 + \gamma_1) + \frac{\gamma_3 + \delta_3 \cdot \lambda_3 \cdot \gamma_1}{1 - \delta_3} < 0 \tag{107}$$

i.e.,

$$\frac{\gamma_3}{1 - \delta_3} + \gamma_1 \cdot \left(1 + \frac{\delta_3 \cdot \lambda_3}{1 - \delta_3}\right) < r_1 \tag{108}$$

or, alternatively,

$$g_{\alpha_1} = g_{x_1} + \gamma_1 < r_1 \tag{109}$$

For $\lambda_3 \approx 0.5, r_1 \approx 0.06, \gamma_1 \approx 0.03, \gamma_3 \approx 0.01$, this implies that either $\delta_3 \lesssim 0.059$ or $-1 < \delta_3 > 1$. For $\gamma_1 \approx 0.02$, this changes to $-1 < \delta_3 \lesssim 0.072$ or $\delta_3 > 1$.

4.7 Results and interpretation

4.7.1 Results

$$g_{x_3} = \frac{\gamma_3 + \delta_3 \lambda_3 \gamma_1}{1 - \delta_3} \tag{110}$$

$$g_{\alpha_3} = g_{x_3} + \gamma_1 = \left[\frac{\gamma_3 + \delta_3 \lambda_3 \gamma_1}{1 - \delta_3}\right] + \gamma_1 \tag{111}$$

$$g_{\alpha_1} = \frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta} \cdot \gamma_1 \tag{112}$$

$$g_{\sigma_1} = \frac{r - \rho}{\eta} - \left(\frac{(1-\eta)(1-\lambda_1)}{\eta} + 1\right) \cdot \gamma_1 - g_{x_3} \tag{113}$$

4.7.2 Example values: $\eta = 1.1, \gamma_1 = 0.03, \delta_3 = 0.5$

$$\begin{cases} \eta = \text{elasticy of spending} = 1.1 \\ \rho = \text{hazard rate} = 0.005 = 0.5\% \\ r_1 = \text{returns above inflation} = 0.06 = 6\% \\ \gamma_1 = \text{change in participant contributions} = 0.03 = 3\% \\ \gamma_3 = \text{change in the difficulty of recruiting} = 0.01 = 1\% \\ w_3 = \text{Average participant contribution per unit of time} = 0.5 \\ \beta_3 = \text{Constant inversely proportional to difficulty of recruiting} = 1,000 \\ \lambda_1 = \text{Coub-Douglas elasticity of direct work and direct spending} = 0.5 \\ \lambda_3 = \text{Coub-Douglas elasticity of movement building} = 0.5 \\ \delta_3 = \text{Elasticity of movement growth} = 0.5 \end{cases} \tag{114}$$

$$\begin{aligned} g_{x_3} &= \frac{\gamma_3 + \delta_3\lambda_3\gamma_1}{1-\delta_3} \\ &= \frac{0.01 + 0.5 \cdot 0.5 \cdot 0.03}{1 - 0.5} \\ &= 0.035 = 3.5\% \end{aligned} \tag{115}$$

$$\begin{aligned} g_{\alpha_3} &= g_{\sigma_3} + g_{x_3} + \gamma_1 = 0 + g_{x_3} + \gamma_1 \\ &= 0.035 + 0.03 \\ &= 0.065 = 6.5\% \end{aligned} \tag{116}$$

$$\begin{aligned} g_{\alpha_1} &= \frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta} \cdot \gamma_1 \\ &= \frac{0.06 - 0.005}{1.1} - \frac{(1-1.1)(1-0.5)}{1.1} \cdot 0.03 \\ &\approx 0.05136 = 5.136\% \end{aligned} \tag{117}$$

$$\begin{aligned} g_{\sigma_1} &= g_{\alpha_1} - g_{x_3} - \gamma_1 \\ &= 0.05136 - 0.035 - 0.03 \\ &= -0.01364 = -1.364\% \end{aligned} \tag{118}$$

4.7.3 Example values: $\eta = 0.9, \gamma_1 = 0.03, \delta_3 = 0.5$

$$\begin{cases} \eta = \text{elasticy of spending} = 0.9 \\ \rho = \text{hazard rate} = 0.005 = 0.5\% \\ r_1 = \text{returns above inflation} = 0.06 = 6\% \\ \gamma_1 = \text{change in participant contributions} = 0.03 = 3\% \\ \gamma_3 = \text{change in the difficulty of recruiting} = 0.01 = 1\% \\ w_3 = \text{Average participant contribution per unit of time} = 0.5 \\ \beta_3 = \text{Constant inversely proportional to difficulty of recruiting} = 1,000 \\ \lambda_1 = \text{Coub-Douglas elasticity of direct work and direct spending} = 0.5 \\ \lambda_3 = \text{Coub-Douglas elasticity of movement building} = 0.5 \\ \delta_3 = \text{Elasticity of movement growth} = 0.5 \end{cases} \tag{119}$$

$$\begin{aligned} g_{x_3} &= \frac{\gamma_3 + \delta_3\lambda_3\gamma_1}{1-\delta_3} \\ &= \frac{0.01 + 0.5 \cdot 0.5 \cdot 0.03}{1 - 0.5} \\ &= 0.035 = 3.5\% \end{aligned} \tag{120}$$

$$\begin{aligned} g_{\alpha_3} &= g_{\sigma_3} + g_{x_3} + \gamma_1 = 0 + g_{x_3} + \gamma_1 \\ &= 0.035 + 0.03 \\ &= 0.065 = 6.5\% \end{aligned} \tag{121}$$

$$\begin{aligned} g_{\alpha_1} &= \frac{r_1 - \rho}{\eta} - \frac{(1-\eta)(1-\lambda_1)}{\eta} \cdot \gamma_1 \\ &= \frac{0.06 - 0.005}{0.9} - \frac{(1-0.9)(1-0.5)}{0.9} \cdot 0.03 \\ &\approx 0.0594 = 5.94\% \end{aligned} \tag{122}$$

$$\begin{aligned} g_{\sigma_1} &= g_{\alpha_1} - g_{x_3} - \gamma_1 \\ &= 0.0594 - 0.035 - 0.03 \\ &= -0.0056 = -0.56\% \end{aligned} \tag{123}$$

4.7.4 Comparison with a rule of thumb allocation

For $\eta = 1.1$**.** Take a rule of thumb allocation, where $\sigma_1 = \sigma_3 = 0.5$, and we spend 1% of our budget per year, which then grows at 5% per year (i.e., $g_{\alpha_1} = g_{\alpha_3} = g_{x_1} = 0.05$).

Let $\lambda_1 = \lambda_3 = 0.5$, and in general let all the variables be as in the $\eta = 1.1$ example. Then the growth rate for x_3 is:

$$g_{x_3} = \gamma_3 + \delta_3 \cdot (\lambda_3 \cdot g_{\alpha_3} + (1 - \lambda_3) \cdot (g_{\sigma_3} + g_{x_3})) \tag{124}$$

$$g_{x_3} = 0.01 + 0.5 \cdot (0.5 \cdot 0.05 + 0.5 \cdot (0 + g_{x_3})) \tag{125}$$

$$g_{x_3} = 0.03 \tag{126}$$

Then consider the growth of U

$$U(x, \alpha) = \frac{(\alpha_1^{\lambda_1} (\sigma_1 x_3)^{1-\lambda_1})^{1-\eta}}{1 - \eta} \tag{127}$$

$$g_U = (1 - \eta) \cdot (\lambda_1 \cdot g_{\alpha_1} + (1 - \lambda_1) \cdot (g_{\sigma_1} + g_{x_3})) \tag{128}$$

$$g_U = (1 - 1.1) \cdot (0.5 \cdot 0.05 + (1 - 0.5) \cdot (0 + 0.03)) = -0.004 \tag{129}$$

In our example, that growth is *smaller*.

$$g_U = (1 - 1.1) \cdot (0.5 \cdot 0.05136 + (1 - 0.5) \cdot (-0.01364 + 0.035)) \approx -0.003636 \tag{130}$$

For $\eta = 0.9$. Using the same reasoning as before

$$g_{x_3} = 0.03 \tag{131}$$

$$g_U = (1 - 0.9) \cdot (0.5 \cdot 0.05 + (1 - 0.5) \cdot (0 + 0.03)) = 0.004 \tag{132}$$

In comparison with

$$g_U = (1-0.9) \cdot (0.5 \cdot 0.0594 + (1-0.5) \cdot (-0.0056 + 0.035)) = 0.00444 \tag{133}$$

So the growth is bigger in our optimal allocation case, when $\eta < 1$! This makes sense.

4.7.5 Interpretation

The model we are considering tilts heavily towards recommending rapid movement growth, and towards spending significant resources to attain it.

Examples of movements which are able to display rapid growth on the order of 2% to 10% a year, sometimes sustainedly, include religions, such as early Christianity, the Later Day Saints Church, or current Islam. But also political and social movements, such as climate activism, neoliberalism, the animal rights movement, the various feminist waves, Bill Gate's *Giving Pledge*, 20th-century communism, the alternative right, utilitarianism, etc.

4.7.6 Sensitivity analysis

To do. Interestingly and counter-intuitively, the multiplicative constants don't end up mattering much (though they do matter for σ_3, the fraction of the movement spent in movement building), it's mostly the growth rates and elasticities.

4.7.7 Practical applications

Limited, because we're finding the balanced growth path, and this can be pretty different from the optimal path at the beginning.

But, I'd wager that in general, marginal value of movement building remains much higher than the marginal value of investment.

4.7.8 Border conditions

In a sense, the space in which we have found a balanced growth path is narrow.

Because of (59), $g_{\sigma_1} \leq 0$:

$$g_{\sigma_1} \leq 0 \tag{134}$$

$$\frac{r-\rho}{\eta} \leq \frac{1-(1-\eta)\lambda_1}{\eta} \cdot \gamma_1 + \frac{\gamma_3 + \delta_3\lambda_3\gamma_1}{1-\delta_3}$$

Further, in (90), we used the approximation $\dot{x}_3 = g_{x_3} \cdot x_3$, respectively $g_{F_3} = g_{\dot{x}_3} = g_{x_3}$, which only holds when $g_{x_3} > 0$, and hence:

$$\frac{\gamma_3 + \delta_3\lambda_3\gamma_1}{1-\delta_3} \geq 0 \tag{135}$$

Similarly, in (59) we made use of $g(\rho \cdot \mu_3 - \dot{\mu}_3) = g(\mu_3)$. When does this hold? Well, let $\mu_3 = exp\{g_{\mu_3} \cdot t\}$, then $\dot{\mu}_3 = g_{\mu_3} \cdot exp\{g_{\mu_3} \cdot t\}$, and $\rho\mu_3 - \dot{\mu}_3 = (\rho - g_{\mu_3}) \cdot \mu_3$, so we want $(\rho - g_{\mu_3}) \geq 0 \implies (\rho \geq g_{\mu_3})$. We can quickly derive g_{μ_3} from $g_{\mu_3} = g_{\mu_1} - g_{F_3} + g_{\alpha_3} = g_{\mu_1} - g_{x_3} + (g_{x_3} + \gamma_1) = g_{\mu_1} + \gamma_1$, so the inequality holds when:

$$g_{\mu_1} + \gamma_1 \leq \rho \tag{136}$$

i.e., with $g_{\mu_1} = \rho - r_1$

$$\gamma_1 \leq r_1 \tag{137}$$

Note that $r_1 \geq \gamma_1$; growth in wages probably can't be higher than overall growth in the long-term. In any case, if g_{x_3} is too high, then the inequality doesn't hold, and in (58), for the variables in a hypothetical balanced growth path, the left side would be negative and the right side positive, so the balanced growth path would be an spurious one, unless both sides converge to 0.

4.7.9 Variable ratios

We can also consider variable ratios. Per (68):

$$\frac{(1-\lambda_1)}{\lambda_1} \cdot \frac{\alpha_1}{\sigma_1} = \frac{(1-\lambda_3)}{\lambda_3} \cdot \frac{\alpha_3}{\sigma_3} \tag{138}$$

This is sensitive to λ_i. In the example above, α_1 and α_3 were increasing, whereas σ_3 tends to a constant, and $\sigma_1 \to 0$. Hence α_3 grows faster than α_1

5 Movement building and research

6 Conclusions

We find that, for a space of plausible parameters, the optimal allocation implies an *asymptotic Ponzi* condition, where, even as the number of movement participants working on either directly generating utility or on research grows with time in absolute terms, they converge to 0% of the total movement size, with most of the movement participants working either on earning money or in movement building. Analogously, even as the amounts of money spent on directly generating utility and on research grow in absolute terms, these amounts also converge to 0% of total yearly spending, with most spending being directed towards movement building.

Additionally, we derive from the Euler equations an easy to apply optimality heuristic, in terms of spending ratios and fractions of movement building allocated to each option, such that not satisfying this heuristic implies that the given movement is not on the optimal allocation path. This heuristic is quite general in the sense that it holds for the models in §3, §4 and §5.

7 References

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Appendices

Appendix A Minimal logarithmic model

A.1 Setup

For the minimal model, returns are logarithmic in the product of research and spending. Research is simply proportional to spending. And at each instant, the previous total budget changes compounds by the interest rate, absent the money spent in either direct work or in research. We ignore movement building.

We are maximizing:

$$V(\vec{\alpha(t)}) = \max_{\vec{\alpha(t)}} \int_0^\infty e^{-\rho t} \cdot U(\vec{x(t)}, \vec{\alpha(t)}) dt \tag{139}$$

For utility and laws of motion:

$$U(x, \alpha) = ln(x_2 \cdot \alpha_1) \tag{140}$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} r_1 x_1 - \alpha_1 - \alpha_2 \\ \alpha_2 \end{bmatrix} \tag{141}$$

under the constraints that

$$\lim_{t \to \infty} x_i \geq 0 \wedge x_2 \geq 0 \wedge \alpha_i \geq 0 \tag{142}$$

With our Hamiltonian standing at:

$$H = ln(x_2 \cdot \alpha_1) + \mu_1 \cdot (r_1 x_1 - \alpha_1 - \alpha_2) + \mu_2 \cdot \alpha_2 \tag{143}$$

and our transversality condition:

$$\lim_{t \to \infty} exp\{-\rho \cdot t\} \cdot x_i \cdot \mu_i = 0 \tag{144}$$

A.2 Optimal path derivation

From constraint (7) on the Hamiltonian:

$$\frac{\partial H}{\partial \alpha_1} = \frac{1}{\alpha_1} - \mu_1 = 0 \implies \alpha_1 = \frac{1}{\mu_1} \tag{145}$$

$$\frac{\partial H}{\partial \alpha_2} = -\mu_1 + \mu_2 = 0 \implies \mu_1 = \mu_2 \tag{146}$$

From constraint (8):

$$-\frac{\partial H}{\partial x_1} = -r_1\mu_1 = \dot{\mu}_1 - \rho\mu_1 \implies \mu_1 = k_1 \cdot exp\{(\rho - r_1)t\} \tag{147}$$

$$-\frac{\partial H}{\partial x_2} = -\frac{1}{x_2} = \dot{\mu}_2 - \rho\mu_2 \tag{148}$$

From (145), (147) we can find the explicit form of α_1:

$$\alpha_1 = \frac{1}{k_1} \cdot exp\{(r_1 - \rho)t\} \tag{149}$$

From (146), (147), (148), we can find the explicit form of x_2:

$$x_2 = \frac{1}{\rho\mu_2 - \dot{\mu}_2} = \frac{1}{k_1 r_1} exp\{(r_1 - \rho)t\} \tag{150}$$

and from (141),(150), a_2:

$$a_2 = \dot{x}_2 = \frac{(r_1 - \rho)}{k_1 r_1} exp\{(r_1 - \rho)t\} = \frac{(1 - \frac{\rho}{r_1})}{k_1} exp\{(r_1 - \rho)t\} \tag{151}$$

Having a_1 and a_2, we can determine x_1 using (141)

$$\dot{x}_1 = r_1 x_1 - \frac{1}{k_1} \cdot exp\{(r_1 - \rho)t\} - \frac{(1 - \frac{\rho}{r_1})}{k_1} exp\{(r_1 - \rho)t\} \tag{152}$$

$$\dot{x}_1 = r_1 x_1 - \frac{2 - \frac{\rho}{r_1}}{k_1} \cdot exp\{(r_1 - \rho)t\} \tag{153}$$

$$x_1 = k_2 \cdot exp\{r_1 t\} + \frac{2 - \frac{\rho}{r_1}}{k_1 \rho} \cdot exp\{(r_1 - \rho)t\} \tag{154}$$

With k_1, k_2 integration constants chosen so that $\vec{x}(0) = \vec{x_0}$. In particular, note that $x_1 < x_{01} \cdot exp\{r_1 t\}$.

A.3 Checking the transversality condition

Consider k_2, the integration constant from solving

$$\dot{x}_1 = r_1 x_1 - \frac{2 - \frac{\rho}{r_1}}{k_1} \cdot exp\{(r_1 - \rho)t\} \tag{155}$$

$$x_1 = k_2 \cdot exp\{r_1 t\} + \frac{2 - \frac{\rho}{r_1}}{k_1 \rho} \cdot exp\{(r_1 - \rho)t\} \tag{156}$$

This constant is uniquely determined by the initial conditions. However, if $k_2 > 0$, then at some point we'll start amassing a fortune which we'll never spend. This is because the k_2 term grows at a rate of r_1, but we're only spending at a rate of $(r_1 - \rho)$. Conversely, if $k_2 < 0$, then at some point we'll go into ever deeper debt, which we never plan to repay.

It is then no coincidence that if $k_2 \neq 0$, our transversality condition (144) doesn't hold, and hence we know the Hamiltonian to output an spurious solution.

$$\begin{aligned}
&\lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot x_1 \cdot \mu_1 \\
&= \lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot \left(k_2 \cdot exp\{r_1 t\} + \frac{2 - \frac{\rho}{r_1}}{k_1 \rho} \cdot exp\{(r_1 - \rho)t\} \right) \\
&\cdot \Big(k_1 \cdot exp\{(\rho - r_1)t\} \Big) \\
&= \begin{cases} 0 \text{ if } k_2 = 0 \\ k_2 \text{ if } k_2 \neq 0 \text{, i.e., the transversality condition doesn't hold} \end{cases}
\end{aligned} \tag{157}$$

We can also check the transversality condition for x_2 and μ_2:

$$\begin{aligned}
&\lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot x_2 \cdot \mu_2 \\
&= \lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot \left(\frac{1}{k_1 r_1} exp\{(r_1 - \rho)t\} \right) \cdot \Big(k_1 \cdot exp\{(\rho - r_1)t\} \Big) \\
&= \lim_{t\to\infty} exp\{-\rho \cdot t\} \cdot \frac{k_1}{k_1 \cdot r_1} = 0
\end{aligned} \tag{158}$$

And this holds, with the possible exception of $k_1 = 0$, i.e., when we start with a fortune of \$0.

A.4 Results and interpretation

When the integration constant k_2 is equal to 0:

$$\alpha_1 = \frac{1}{k_1} \cdot exp\{(r_1 - \rho)t\} \tag{159}$$

$$a_2 = \frac{(1 - \frac{\rho}{r_1})}{k_1} exp\{(r_1 - \rho)t\} \tag{160}$$

$$x_1 = \cancel{k_2 \cdot exp\{r_1 t\}} + \frac{2 - \frac{\rho}{r_1}}{k_1 \rho} \cdot exp\{(r_1 - \rho)t\} \tag{161}$$

$$x_2 = \frac{1}{k_1 r_1} exp\{(r_1 - \rho)t\} \tag{162}$$

In this simple model, both investment in research and direct altruistic spending would grow at the same exponential rate $(r_1 - \rho)$, as long as the rate of returns outpaces the hazard rate ρ.

The shape of research spending differs from that of direct altruistic spending by a multiplicative factor of $(1 - \frac{\rho}{r_1})$, meaning that as the return rate approximates the hazard rate, one would stop spending much on research, though not on direct giving. Commonly, however, $\rho << r_1$, so this multiplicative factor will be close to one.

In particular, for $r = 0.07 = 7\%, \rho = 0.005 = 0.5\%$,

$$\frac{\alpha_1}{\alpha_2} = \frac{1}{1 - \frac{\rho}{r_1}} = \frac{1}{1 - \frac{0.005}{0.07}} = \frac{14}{13} \approx 1.0769 \tag{163}$$

Lastly, total accumulated capital also grows in expectation, although at a lower pace than if it was left to compound alone.